Topological Open p-Branes

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By exploiting the BV quantization of topological bosonic open membrane, we argue that flat 3-form C-field leads to deformations of the algebras of multi-vectors on the Dirichlet-brane world-volume as 2-algebras. This would shed some new light on geometry of M-theory 5-brane and associated decoupled theories. We show that, in general, topological open p-brane has a structure of $(p+1)$-algebra in the bulk, while a structure of p-algebra in the boundary. The bulk/boundary correspondences are exactly as of the generalized Deligne conjecture (a theorem of Kontsevich) in the algebraic world of p-algebras. It also imply that the algebras of quantum observables of $(p-1)$-brane are "close to" the algebras of its classical observables as p-algebras. We interpret above as deformation quantization of $(p-1)$-brane, generalizing the $p=1$ case. We argue that there is such quantization based on the direct relation between BV master equation and Ward identity of the bulk topological theory. The path integral of the theory will lead to the explicit formula. We also discuss some applications to topological strings and conjecture that the homological mirror symmetry has further generalizations to the categories of p-algebras.

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1 Introduction

The discovery of D-branes – extended objects carrying RR charge, has greatly enhanced our understanding of string theory [65]. D-branes can be realized as certain Dirichlet boundary condition of the fundamental open string. One can also say that the fundamental open strings describe excitations of the D-brane. The low energy dynamics of a D-brane is described by the maximally supersymmetric Yang-Mills (SYM) theory on the D-brane worldvolume. The open string can naturally be coupled to flat NS 2-form B-fields. It is by now well-known that there are suitable decoupling limits of the bulk degrees of freedom, and the dynamics of the D-brane worldvolume is described eitherby non-commutative SYM or by the non-commutative open string [73, 72, 33]. The basic picture is that the B-field induces a non-commutative deformation of the algebra $\mathcal{O}(X)$ of functions on the D-brane world-volume X; it makes the world-volume non-commutative [20, 66, 73].

For the non-commutativity the supersymmetry and the metric are secondary. We can consider open bosonic strings in an arbitrary number of Euclidean dimensions coupled only with the B-field (we may allow a Poisson bi-vector in general). Then the problem becomes equivalent to the quantization of the boundary particle theory, which has the "D-brane" world-volume as its classical phase space [17]. This consideration leads to the path integral derivation of the celebrated solution of deformation quantization by Kontsevich [50]. A crucial physical insight in the above approach is that the physical consistency of the bulk open string theory implies associativity of the non-commutative deformation of the algebra of functions on the D-brane world-volume [17]. A simple generalization of the above leads to a physical proof of the formality theorem of Kontsevich. This implies a deep connection between open strings and the world of associative algebras.

Strominger [77] and Townsend [82] showed that the M theory 5-brane can be interpreted as a D-brane of the open super-membrane and all D-branes of Type IIA string can be obtained by S^1-compactification. They also argued that the boundary dynamics of the above system is controlled by a six-dimensional self-dual string [96]. The open membrane can naturally be coupled with a flat 3-form C-field. The presence of the M5-brane requires self-duality for the parallel C-field. Recently the authors of [3, 73, 11, 43, 34] showed that there is suitable decoupling limit such that the bulk theory becomes topological and only the modes in the brane are left. The resulting theory is now called OM theory, which is related to other decoupled theories by a web of dualities [34]. An open question is the algebraic or geometrical meaning of turning on such a background, the resulting boundary dynamics,

etc.

The main purpose of this paper is to uncover the basic picture on the role of C-field in more mundane situations. For this it is suffices to study the bosonic open membrane coupled with the C-field only. We call the resulting theory the topological open membrane theory. The topological open membrane makes sense in arbitrary dimensions, as does the topological open string. Actually we will start from one further step back by considering the open membrane without background. Then we interpret the topological open membrane theory as a certain deformation of the theory without background. The theory without background will tell us that the underlying algebraic structure of the boundary string theory is the Gerstenhaber algebra (G-algebra in short [30]) of polyvectors on X. More appropriately it is the algebra $\mathcal{O}(\Pi T^*X)$ of functions on the superspace ΠT^*X, which is the total space of the cotangent bundle over X after a parity change of the fiber. Then quantum consistency of the theory requires that the C-field must be flat, which corresponds to the infinitesimal deformation of the above G-algebra as a strongly homotopy G-algebra (a G_∞-algebra in short [79, 81] or 2-algebra [50]). We call the resulting algebra the 2-algebra of X. Then one may specialize to the 6-dimensional case and consider the deformation by a self-dual C-field only. We may call this the self-dual 2-algebra of six dimensions X. An interesting point of OM theory is that it requires a non-vanishing constant self-dual C-field [34]. Thus the theory from the beginning should involve the deformed 2-algebra or self-dual 2-algebra of X. We will leave the detailed study of path integrals (deformed algebra) and applications to physics for a future publication [37].

Our approach also has a natural generalization to higher dimensional topological open p-branes. We shall see that open p-branes have a deep connection with the world of p-algebras. The bulk and the boundary correspondence of open p-brane theory follows exactly the generalized Deligne conjecture involving $(p + 1)$ and p-algebra [51]. The crucial tools for our approach are the Feynman path integrals á la BV quantization [8, 89, 67, 68, 91, 2, 69]. We will also discuss some applications to the homological mirror conjecture [49]. We should mention that the 2-algebra is not an alien to string theory. It already appeared in the closed and open-closed string field theories of Zwiebach [102, 103], VOA, TCFT and $D = 2$ string theory [100, 85, 101, 58, 14, 64, 32] (see also [76] for a nice review).

Now we begin a rather detailed introduction or sketch of our program, treating all topological open p-branes uniformly and emphasizing the more mathematical side of our story.

A sketch of our program

The basic idea of deformation quantization is that the algebra of observables in quantum mechanics is close, as an associative algebra, to the commutative algebra of functions on the classical phase space [9, 50]. Thus the program reduces to finding formal deformations of the commutative algebra along non-commutative directions as an associative algebra. This is realized as deformations of the usual products of functions to star products, whose associativity automatically implies that an infinitesimal should be a Poisson bi-vector. A surprise of Kontsevich's result is that the quantization of the particle somehow requires open string theory [50]. Catteneo and Felder showed that Kontsevich's formula is the perturbative expansion of the path integral of bosonic topological open string theory [17].

A novelty of this approach is that the problem of deformation quantization of particles (thus the deformation of an associative algebra) is equated to quantum consistency of the bosonic open string theory. This maps the set of equivalence classes of Poisson structures on the target space X to the set of equivalence classes of deformations of the open string theory satisfying the BV master equation. Then the BV master equation automatically implies, via the Ward identity, that a suitable path integral of the theory on the disk defines a bijection from the above equivalence class to the set of isomorphism classes of associative star products. In general this approach leads to a string theoretic derivation of the formality theorem of Kontsevich. The unifying mathematical notion behind the above correspondences is the operads of little intervals and associated Swiss-Cheese operads, which relate associative algebras with 2-algebras (Deligne conjecture) [79, 80, 51]. On the other hand the Swiss-Cheese operad [86] is closely related to the tree level open-closed string field theory of Zwiebach [103]. Actually the path integral approach shows that the Deligne conjecture is just the bulk/boundary correspondence. Recently Hofman and Ma discussed those interrelations in a more general class of topological open-closed string theory [38] (see also [56, 63] on some recent development on topological open-closed string theory).

We shall see that the topological open p-brane for any $p > 0$ is closely related with the world of $(p+1)$ and p-algebras. This generalizes the relation in the $p = 1$ case, where 1-algebra is another name for associative algebra. We may use topological open p-brane theory to define deformation quantization of the boundary closed bosonic $(p-1)$-brane. We shall see that the problem is equivalent to the problem of deformation of p-algebra as a p-algebra. Our basic tool is the method of BV quantization for Feynman

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path integrals. Here we sketch the general principle of our program.

Bosonic open p-brane theory is a theory of maps

$$\phi : N_p \to X,$$

where N_p is a $(p + 1)$-dimensional manifold with boundary, regarded as the open p-brane world volume, and X is the target space. The bosonic topological open p-brane coupled with a $(p + 1)$-form c in X (open p-brane in a closed NS p-brane background) is described by the action functional

$$I' = \int_{N_p} \phi^*(c) + \int_{\partial N_p} \mathcal{V} \tag{1.1}$$

where $\mathcal{V}$ denotes a possible boundary interaction. We will regard the above theory as describing deformations of bosonic open p-brane theory *without background* defined by certain action functional I_o being first order in derivatives and invariant under affine transformations of X. After BV quantization one may obtain the BV master action functional $\boldsymbol{S}_o$ of I_o. Then we examine consistent deformations, modulo equivalence, of $\boldsymbol{S}_o$ using the BV master equation. The resulting theory $\boldsymbol{S}'$ will be identified with the BV quantization of the theory I' if the deformation preserves the ghost number symmetry.

In general the undeformed theory $\boldsymbol{S}_o$ tells us which mathematical structure (associated with the boundary degrees) we want to deform. It can be determined by correlation functions of observables inserted on the boundary ∂N_p. Now the bulk deformation term in the action functional tells us how to deform the mathematical structure by specifying only the infinitesimal deformation. And the perturbative expansion with the bulk term determines the deformation of the mathematical structure to all orders. Note that the consistent bulk deformation term is determined by the BV master equation, while the correlation functions should satisfy the BV Ward identity of the theory, which is a direct consequence of the BV master equation.

It turns out that the BV quantized topological open p-brane theory is a theory of maps

$$\phi : \Pi T N_p \to M_p(X) \tag{1.2}$$

between two superspaces $\Pi T N_p$ and $M_p(X)$ associated with N_p and X, respectively.

The superspace $\Pi T N_p$ is the total space of the tangent bundle of N_p after parity change of the fiber. We denote a set of local coordinates on $\Pi T N_p$ by $(\{x^\mu\}|\{\theta^\mu\})$, $\mu = 1, \ldots, p + 1$, where θ^μ are odd constants. We introduce the ghost number or degree $U \in \mathbb{Z}$. We assign $U = 1$ to θ^μ.

The target superspace $M_p(X)$ of the p-brane for $p \geq 1$ can most easily be described recursively. M_p for $p \geq 1$ is the total space of the twisted by $[p]$ cotangent bundle $T^*[p]M_{p-1}$ over M_{p-1} and $M_0 = X$ is the target space X. For example

$$
\begin{aligned}
M_1 &= T^*[1]X \equiv \Pi T^*X, \\
M_2 &= T^*[2]M_1 = T^*[2]\left(\Pi T^*X\right), \\
M_3 &= T^*[3]M_2 = T^*[3]\left(T^*[2]\left(\Pi T^*X\right)\right),
\end{aligned}
\tag{1.3}
$$

etc. Note that the base space M_{p-1} of the target superspace M_p of the p-brane is the target superspace of the $(p-1)$-brane. Physically for open the p-brane, M_{p-1} corresponds to the target superspace of boundary the $(p-1)$-brane. The iterative nature of target superspace is due to the degeneracy of the first order formalism, which requires to introduce ghosts for ghosts etc.

Now we explain the notation $T^*[p]M_{p-1}$. We introduce the following set of local coordinates $(base|fiber)$ on $M_p \to M_{p-1}$;

$$
(\{q^\alpha\}|\{p_\alpha\}),
\tag{1.4}
$$

where $\alpha = 1, \ldots, 2^{p-1} \times \dim(X)$. We assign ghost number U or degree of such coordinates by the formula

$$
U(q^\alpha) + U(p_\alpha) = p, \qquad U(q^\alpha) \geq 0, \quad U(p_\alpha) \geq 1.
\tag{1.5}
$$

Thus the twisted by $[p]$ cotangent bundle $T^*[p]M_{p-1}$ over M_{p-1} is the cotangent bundle over M_{p-1} with the above assignment of ghost number. A coordinate is commuting or even if the ghost number is even. A coordinate is anti-commuting or odd if the ghost number is odd. An index α can be either a tangent or a cotangent index of X and indices for q^α and p_β should be (up, down) or (down,up) for $\alpha = \beta$. The ghost number of various coordinates can be determined recursively by assigning $U = 0$ to local coordinates on $M_0 = X$. For instance the ghost numbers of, say, base coordinates q^α of M_p can be different in general for each α. We also note that $M_p(X)$ for $p \geq 2$ can be identified with the total space of $(p-1)$th iterated supertangent bundle over ΠT^*X, i.e., $M_p(X) \simeq \Pi T(\Pi T(\ldots(\Pi T(\Pi T^*X))\ldots))$.

We describe a map $\boldsymbol{\phi} : \Pi T N_p \to M_p$ locally by local coordinates on M_p

$$
(\boldsymbol{q}^\alpha, \boldsymbol{p}_\alpha) := (q^\alpha(x^\mu, \theta^\mu), p_a(x^\mu, \theta^\mu))
\tag{1.6}
$$

which are functions on $\Pi T N_p$. The superfields $(\boldsymbol{q}^\alpha, \boldsymbol{p}_\alpha)$ combine all the "fields" and "anti-fields" of the theory. The assignment of ghost numbers (1.5) are a consequence of BV quantization.

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We note that the target superspace $M_p(X)$, for $p \geq 1$, always has the following non-degenerate canonical symplectic form ω_p for any manifold X as the total space of the (twisted by $[p]$) cotangent bundle over M_{p-1};

$$\omega_p = dp_\alpha \wedge dq^\alpha. \tag{1.7}$$

The symplectic form ω_p carries degree $U = p$. The parity of ω_p is the same as the parity of p. Now the degree $U = p$ symplectic structure ω_p on M_p defines a degree $U = -p$ (odd or even) graded Poisson bracket $[.,.]_{p+1}$ on functions on M_p. The BV bracket $(.,.)_{BV}$ of the p-brane theory is an odd Poisson bracket with degree $U = 1$ among local functions on the space $\mathcal{A}$ of all maps $\phi : \Pi T N_p \to M_p$. The corresponding odd symplectic form ω with degree $U = -1$ on $\mathcal{A}$ originates from a degree $U = p$ (odd or even as the same parity of p) symplectic form ω_p on the (super)-target space M_p by the formula

$$\omega := \int_{N_p} d^{p+1}\theta \, \phi^*(\omega_p). \tag{1.8}$$

The super-integral shifts the degree U by $(-p-1)$. The above considerations lead us to the following crucial relation

$$\left(\int_{N_p} d^{p+1}\theta \, \phi^*(\gamma), \int_{N_p} d^{p+1}\theta \, \phi^*(\gamma) \right)_{BV} = \int_{N_p} d^{p+1}\theta \, \phi^*([\gamma, \gamma]_{p+1}), \tag{1.9}$$

where γ is a local function on M_p.

The BV action functional $\boldsymbol{S}_o$ of the topological open p-brane without background is defined as follows

$$\boldsymbol{S}_o = \int_{N_p} d^{p+1}\theta \left(\boldsymbol{p}_\alpha D\boldsymbol{q}^\alpha + \phi^* \left(h(q^\alpha, p_a) \right) \right), \tag{1.10}$$

where $D = \theta^\mu \partial_\mu$ and $h(q^\alpha, p_a)$ is a degree $U = p + 1$ function on M_p. The function $h(q^\alpha, p_a)$ is invariant under affine transformations on X and constant on X. It satisfies

$$[h, h]_{p+1} = 0, \qquad h(q^\alpha, p_\alpha)|_{M_{p-1}} = 0. \tag{1.11}$$

Thus h generates a differential Q_o with $U = 1$ and $Q_o^2 = 0$ via the bracket,

$$[h, \ldots]_{p+1} = Q_o. \tag{1.12}$$

Futhermore $Q_o|_{M_{p-1}} = 0$. It turns out that $h = 0$ if $p = 1$ and h for general p can be determined recursively. Note also that

$$\left(\int_{N_d} d^{d+1}\theta \, (\boldsymbol{p}_\alpha D\boldsymbol{q}^\alpha), \ldots \right)_{BV} = D. \tag{1.13}$$

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The BV BRST charge $\boldsymbol{Q}_o$ carrying $U = 1$ corresponds to an odd Hamiltonian vector of $\boldsymbol{S}_o$ on the space $\mathcal{A}$ of all fields, i.e., $\boldsymbol{Q}_o = (\boldsymbol{S}_o, \ldots)_{BV}$. Then we obtain another crucial relation;

$$\boldsymbol{Q}_o = D + \phi^*(Q_o). \tag{1.14}$$

Combining (1.9), (1.9), and (1.14), we see that the action functional satisfies the quantum master equation if the boundary conditions are such that $\boldsymbol{p}_\alpha(x) = 0$ "in directions tangent" to ∂N_p for $x \in \partial N_p$.[1]

Now we describe the possible bulk deformations. We consider any function (or sum of functions in general) $\gamma(q^\alpha, p_\alpha)$ of M_p, whose degree $U = |\gamma|$ has the same parity as $p + 1$. The action functional $\boldsymbol{S}_\gamma$ deformed by γ is given by

$$\boldsymbol{S}_\gamma = \boldsymbol{S}_o + \int_{N_p} d^{p+1}\theta \; \phi^* \left(\gamma \left(p^\alpha, q_\alpha \right) \right). \tag{1.15}$$

The above corresponds to an even function on $\mathcal{A}_p$. Combining (1.9) and (1.13), we see that the deformed action functional satisfies the master equation $(\boldsymbol{S}_\gamma, \boldsymbol{S}_\gamma)_{BV} = 0$[2] if and only if

$$\int_{\partial N_p} d^p\theta \; \phi^*(\gamma(q^\alpha, p_\alpha) = 0, \tag{1.16}$$
$$[h + \gamma, h + \gamma]_{p+1} = 0.$$

Note that the first condition gives a reason to call $\int_{N_p} d^{p+1}\theta \; \phi^* (\gamma)$ the bulk term. Now the boundary conditions and the definition (1.12) imply that the above conditions are equivalent to

$$\gamma|_{M_{p-1}} = 0, \tag{1.17}$$

Thus the set of equivalence classes of bulk deformations of the topological p-brane theory, satisfying the BV master equation, is isomorphic to the set of equivalence classes of solutions of the Maurer-Cartan (MC) equation for functions γ on M_{p+1} satisfying the condition $\gamma|_{M_{p-1}} = 0$. We denote the Hamiltonian vector of $h + \gamma$ by Q_γ;

$$Q_\gamma := [h + \gamma, \ldots]_{p+1}, \tag{1.18}$$

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[1] It means that the worldvolume (N_p) scalar components $p_\alpha(x^\mu)$ of $\boldsymbol{p}_\alpha$ vanish at the boundary, the vector components in directions tangent to ∂N_p vanish at the boundary, etc.

[2] In the present model the classical master equation also implies the quantum master equation.

which satisfies $Q_\gamma^2 = 0$ due to the second equation in (1.16). We note that the restriction $Q_\gamma|_{M_{p-1}}$ of Q_γ to the base space M_{p-1} corresponds to a first order differential acting on functions on the base space M_{p-1}.

The BRST charge $\boldsymbol{Q}_\gamma \equiv (\boldsymbol{S}_\gamma, ...)$ or the odd Hamiltonian vector of $\boldsymbol{S}_\gamma$ is given by

$$\boldsymbol{Q}_\gamma = D + \phi^*(Q_\gamma), \tag{1.19}$$

which satisfies $\boldsymbol{Q}_\gamma^2 = 0$ due to the master equation. This implies that, for any function κ on M_p satisfying $Q_\gamma \kappa = 0$, we have

$$\boldsymbol{Q}_\gamma \phi^*(\kappa) = D \phi^*(\kappa). \tag{1.20}$$

This gives us so called the descent equations. Related to the above we note that the action functional $\boldsymbol{S}_\gamma$ has another fermionic symmetry generated by an odd vector $\mathcal{K}_\mu = -\frac{\partial}{\partial\theta^\mu}$ with $U = -1$ since it is written as a superspace integral. Then (1.14) implies that

$$\{\boldsymbol{Q}_\gamma, \boldsymbol{Q}_\gamma\} = 0, \qquad \{\boldsymbol{Q}_\gamma, \mathcal{K}_\mu\} = -\partial_\mu, \qquad \{\mathcal{K}_\mu, \mathcal{K}_\nu\} = 0. \tag{1.21}$$

Now we turn to the boundary interactions and boundary observables. The boundary interaction due to the boundary conditions is given by a certain local functional $\mathcal{V}(\boldsymbol{q}^\alpha, D\boldsymbol{q}^\alpha)$ of $\boldsymbol{q}^\alpha$ and $D\boldsymbol{q}^\alpha$;

$$\boldsymbol{S}'_o = \boldsymbol{S}_o + \int_{\partial N_p} d^d\theta \, \mathcal{V}\left(\boldsymbol{q}^\alpha, D\boldsymbol{q}^\alpha\right). \tag{1.22}$$

The above action functional satisfies the quantum master equation for any such $\mathcal{V}$ satisfying $(\phi^*(Q_\gamma))\mathcal{V} = 0$. Clearly $\mathcal{V}$ should have $U = p$ to preserve the ghost number symmetry. We consider N_p as a $(p+1)$-dimensional disk with boundary $\partial N_p = S^p$. On the boundary we have $n+1$ punctures $x_0, \ldots x_n$. We also consider S_i^{p-1} surrounding a puncture. We consider a local function $f(q^\alpha)$ of the base of $M_p \to M_{p-1}$ satisfying $Q_\gamma|_{M_{p-1}} f(q_\alpha) = 0$. Now we let $\boldsymbol{f} := f(\boldsymbol{q}_\alpha) := f(q_\alpha(x^\mu, \theta^\mu))$ be the corresponding function of superfields $\boldsymbol{q}^\alpha$. The descent equation (1.20) implies that $\boldsymbol{Q}_\gamma \boldsymbol{f} = D\boldsymbol{f}$ and we obtain the following non-trivial BV observables

$$\begin{aligned}
\mathcal{O}_f^0(x_i) &= f(x_i), \\
\mathcal{O}_f^{(p-1)}(x_i) &= \oint_{S_i^{p-1}} d^{p-1}\theta \, \boldsymbol{f}, \\
\mathcal{O}_f^{(p)} &= \int_{\partial N_p} d^p\theta \, \boldsymbol{f}.
\end{aligned} \tag{1.23}$$

The last one above may be regarded as part of the boundary interaction.

Now we turn to the role of the path integral. For simplicity we ignore boundary interactions and consider the action functional $\boldsymbol{S_\gamma}$

$$\boldsymbol{S_\gamma} = \int_{N_d} d^{d+1}\theta\, (\boldsymbol{p}_\alpha D\boldsymbol{q}^\alpha) + \int_{N_d} d^{d+1}\theta\, \Big(h(\boldsymbol{q}^\alpha, \boldsymbol{p}_a) + \gamma(\boldsymbol{q}^\alpha, \boldsymbol{p}_a) \Big). \quad (1.24)$$

The first term is the kinetic term and the remaining terms may be regarded as "interaction" terms. Note that the interaction terms are coming from the function $(h + \gamma)$ on M_p with $(h + \gamma)|_{M_{p-1}} = 0$, which we called bulk terms. After a suitable gauge fixing one may evaluate correlation functions of observables supported on the boundary ∂N_p using perturbation theory. Note that such observables originate from functions on the base space $M_{p-1}(X)$ of $M_p \to M_{p-1}(X)$.

Now we introduce the following definition

> We call the algebra $\mathcal{O}(M_p(X))$ of functions on $M_p(X)$ with the bracket $[.,.]_{p+1}$, and ordinary (super-commutative and associative) product the classical $(p+1)$-algebra $Cl_{p+1}(X)$ of X. Thus, by definition, the classical p algebra $Cl_p(X)$ of X is the algebra $\mathcal{O}(M_{p-1}(X))$ of functions on the base space of $M_p \to M_{p-1}$.

Note that the classical 1-algebra $Cl_1(X)$ ($p = 0$) is the algebra $\mathcal{O}(X)$ of functions on X without a bracket (since we do not have ω_0). The classical 2-algebra $Cl_1(X)$ ($p = 1$) is the algebra of polyvectors on X with the wedge product and Schouten-Nijenhuis bracket.

Combining altogether, we showed that

> The BV structure of the topological bosonic open p-brane theory originates from the classical $(p + 1)$-algebra $Cl_{p+1}(X) = \mathcal{O}(M_p(X))$. A BV master action functional of the theory is determined by a bulk term associated with $(h + \gamma)$, which is a function on M_p with $(h + \gamma)|_{M_{p-1}} = 0$ solving the MC equation of $Cl_{p+1}(X)$. The classical algebra of observables in the boundary is the classical p-algebra $Cl_p(X)$, which is the algebra $\mathcal{O}(M_{p-1}(X))$ of functions on the base space M_{p-1} of $M_p \to M_{p-1}$.

The perturbative expansion of the theory can be viewed as a certain morphism $Cl_{p+1}(X) \to Qh_{p+1}(X)$ satisfying suitable Ward identities controlled by the bulk theory. We call $Qh_{p+1}(X)$ the quantum $(p + 1)$ algebra. Note that the Ward identity is a direct consequence of the BV master equation.

Thus the set of equivalence classes of dolutions to the BV master equation is isomorphic to the set of equivalence classes of of solutions to the Ward identity. This implies that the morphism is a quasi-isomorphism. Now the quantum algebra of observables (defined by the correlation functions) should be a deformation of the classical p-algebra $Cl_p(X)$. We call the resulting algebra of quantum observables the quantum p-algebra $Qh_p(X)$. This is again controlled by the bulk Ward identity.

It turns out that the classical p-algebra $Cl_p(X)$ is an example of the so called cohomological p-algebra $H^*(A_p)$ [31, 50]. Kontsevich defined a p-algebra as an algebra over the operad $\mathtt{Chains}(\mathtt{C}_p)$, where $\mathtt{C}_p$ is the p-dimensional little disk operad. According to Kontsevich an algebra over the cohomology $H_*(\mathtt{C}_p)$ (cohomological p-algebra in short) is a twisted Gerstenhaber algebra with Lie bracket with degree $1-p$ for $p=2k$, where k is a positive integer, and a twisted Poisson algebra with Lie bracket with degree $1-p$ for $p=2k+1$, both with the commutative associative product of degree 0 and zero differential [51]. This agrees with our definition of a classical p-algebra $Cl_p(X)$.[3]

The generalized Deligne conjecture says that for every p-algebra there exists an universal $p+1$ algebra acting on it. For example there is a structure of $(p+1)$ algebra on the generalized Hochschild complex $\mathtt{Hoch}(A_p)$ of a p-algebra A_p [51, 80]. The picture is that there is a $(p+1)$-algebra controlling deformations of a p-algebra as a p-algebra. Kontsevich, generalizing Tarmarkin [79], proved the above conjecture as well as the formality of p-algebra, i.e., $\mathtt{Chains}(\mathtt{C}_p) \otimes \mathbb{R}$ is quasi-isomorphic to its cohomology $H_*(\mathtt{C}_p) \otimes \mathbb{R}$ endowed with zero differential. In particular the two sets of equivalence classes of solutions to the Maurer-Cartan (MC) equations on $\mathtt{Chains}(\mathtt{C}_p) \otimes \mathbb{R}$ and $H_*(\mathtt{C}_p) \otimes \mathbb{R}$ are isomorphic. Kontsevich suggested that the generalized Deligne conjecture seems to be related to quantum field theories with boundaries. He also conjectured the existence of a structure of p-algebra on p-dimensional conformal field theories [51].

It is amusing to see that all the above is beautifully realized in the topological open p-brane theory.[4] Recall that the bulk theory is determined by a cohomological $(p+1)$-algebra as the algebra of functions on $M_p(X)$, while the boundary observables are associated with a cohomological p-algebra as the algebra of functions on the base space M_{p-1} of $M_p \to M_{p-1}$. Note that the algebra of polynomials $\mathbb{R}[\{q^\alpha\}]$ – the polynomials in the coordinates

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[3]Note, however, that we generally have a differential.

[4]We note that such a possibility was considered before [38, 21], though without actual realizations.

of the base space $M_{p-1}(X)$ – can be viewed as a cohomological p-algebra without differential. It is easy to see (following sect.3.4. of [50]) that the Hochschild cohomology of the above cohomological p-algebra is the algebra $\mathbb{R}[\{q^\alpha\}, \{p^\alpha\}]$ – the polynomials in the coordinates of the total space M_p, without bracket. Note also that our assignment of ghost numbers is consistent with [50]. One may endow $\mathbb{R}[\{q^\alpha\}]$ with a bracket. Then the the Hochschild cohomology should have a differential. It turns out that the differential is exactly the differential Q_o associated with the term h that appeared in the definition of the open p-brane without background. In fact the term $\int_{N_p} d^{p+1}\theta \, \phi^*(h)$ in the action functional is responsible for the appearance of the bracket of the p-algebra by correlation functions.

It is also not difficult to see the appearance of $\texttt{Hoch}(A_p)$ and the structure of $(p + 1)$-algebra. Applying a theorem in [51] it is easy to see that the Hochschild cohomology $H^*(\texttt{Hoch}(Cl_p(X)))$ of the classical algebra $Cl_p(X)$ is the classical $(p+1)$-algebra $Cl_{p+1}(X)$. Thus a bulk "interaction" term of the p-brane theory is an element of $H^*(\texttt{Hoch}(Cl_p(X)))$.[5] The bulk "interaction" term (in the path integral) generates elements of the Hochschild complex $\texttt{Hoch}(Cl_p(X))$ of the classical p-algebra $Cl_p(X)$ by perturbative expansions.

Thus all together we may conclude that the path integral of the topological open p-brane serves as a morphism between the two $(p+1)$-algebras associated with the target space X; $H^*(\texttt{Hoch}(Cl_p(X))$ and $\texttt{Hoch}(Cl_p(X))$. As we argued before the morphism must be a quasi-isomorphism due to the BV master equation of the topological open p-brane theory. One should check this by working out the BV Ward identity of the theory by carefully working out the compactification of the moduli space, related with the higher dimensional Swiss-Cheese operads. Furthermore the Ward identity will tell us that the quantum algebra $Qh_p(X)$ of observables has the structure of a p-algebra. This motivates us to define the deformation quantization of the $(p − 1)$-brane as the deformation of the algebra $\mathcal{O}(M_{p-1}(X)) \equiv Cl_p(X)$ of functions on $M_{p-1}(X)$ as a p-algebra.

We note the path integral derivation of the formality theorem for the $p = 1$ case goes along the same lines as the WDDV equation [95, 23], and as that of 2D strings [100, 85] involving compactification of the moduli space. Similarly the topological open p-brane is determined essentially by the bulk BV master action functional involving $Cl_{p+1}(X)$, boundary observables involving $Cl_p(X)$ and the moduli space of disks with boundary punctures. The

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[5]It seems to be more natural to modify Kontsevich's definition (Sect. 3.4 in [50]). In our case $H^*(\texttt{Hoch}(Cl_p(X)))$ is an algebra of functions on $M_{p+1}(X)$ but with an additional condition that such a function should vanish after restriction to $M_{p-1}(X)$.

rest is determined by the Feynman path integral. It is highly unlikely that the quantum algebra $Cl_p(X)$ is not a p-algebra defined by Kontsevich based on the same data [51]. Now there are several mathematical proofs of the generalized Deligne conjecture of formality of $p + 1$ algebra [79, 50, 81, 52]. However none of those proofs seem to give an explicit quasi-isomorphism except for the $p = 1$ case [49]. The topological open p-brane theory will lead to such an explicit formula. The details will appear elsewhere [37]. It will be also interesting to see if the path integrals of topological open p-brane "confirm" another conjecture of Kontsevich on the action of motivic Galois group [51].

We like to mention that our approach is quite similar to that of Witten in his attempt to formulate background independent open string field theory, suggesting a generalization for the closed string case [94]. It seems to be also closely related to the use of Chern-Simons theory in 3 dimensions on rational conformal field theory and vise versa [98]. The model for $p = 2$ is related with the so called extended or BV Chern-Simons theory [48, 5, 2, 69] and the BV quantized higher dimensional BF theory [42, 18]. The interplay between bulk and boundary obviously reminds of the Maldacena conjecture [60, 36, 99], which is perhaps a purely algebraic (or geometrical) remnant. There is long history on deep relations between open string theory and 1-algebras (A_∞-algebras [74]). A central example is the open string field theory based on a non-commutative and strictly associative algebra [88] and its generalization up to homotopy [29]. The associativity up to homotopy allows one to construct an action functional of open string field theory satisfying the Batalin-Vilkovisky (BV in short) master equation, therefore admitting a consistent quantization. Similar structures also appear as Fukaya's A_∞-category [25] in the open string version of the so called A model of the topological sigma model [93]. The open-closed string field theory [103] is related to the algebra over Swiss-Cheese operads, while the closed string field theory is based on the L^∞ part of the homotopy 2-algebra [102, 45, 46]. It is also natural to expect that the homological mirror conjecture [49] can eventually be the physical equivalence of open-closed string field theory. It seems to be also reasonable to suspect that one may define open-closed p-brane field theory using higher dimensional Swiss-Cheese operads with additional decoration as in the $p = 1$ case. Actually the above is true only for the string tree level. The string field theory is based on a certain loop generalization of the above. For higher dimensional branes it is practically impossible to consider any "loops" as summing over all topologies and geometries. We may content with the theory defined on tree level.

The structure of this paper

This paper is organized as follows. In Sect. 2 we review the BV quantization method, emphasizing the underlying super-geometrical structure and relations with deformation theory. Then we reformulate the Catteno-Felder model and the A model in a language suitable for our purpose. This section will set up notations and our general strategy. In Sect. 3 we study BV quantization of the topological open membrane. This is a detailed example of the general structure discussed in the introduction for $p = 2$. We also present the leading deformations of $Cl_2(X)$. We also discuss different choices of boundary conditions and their implications to possible generalizations of homological mirror symmetry to the category of homotopy 2-algebras (open membrane). In Sect. 4 we return to the $p = 1$ case (string). We introduce an unified topological sigma model, which has the A, B and Catteneo-Felder models as special limits. We construct an extended B model parametrized by the extended moduli space of complex structures and show how the non-commutativity appears for the open string case. We briefly discuss applications of the extended B model to the homological mirror conjecture. We also conjecture that the homological mirror conjecture can be generalized to the category of any p-algebra of X or the physical equivalence of any topological open p-brane theory.

2 Preliminary

In this section we begin by reviewing the method of BV quantization. For details we refer to [8, 91, 67, 68, 94, 2]. We compare it with the modern deformation theory and argue, as a general statement, that deformation problems of certain mathematical structures may be represented as a BV quantization problem of a suitable quantum field theory and vice versa. Then we reconsider the formulation of Kontsevich-Catteneo-Felder as a deformation problem of bosonic string theory. We also discuss BV quantization of the A model, generalizing [2], for a later purpose. We do not assume any originality in this section, but perhaps some new interpretations.

2.1 BV quantization and deformation theory

A path integral is a formal integral of certain observables over the space of all fields of a classical theory weighted by the exponential of the classical action functional I. An observable of the theory is a function on the space

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of all fields invariant under the symmetries of the classical action functional. Consequently one needs to mod out volume of the orbit of symmetry group. Thus we need to construct a "well-defined" quotient measure for the path integral. The BRST-BV quantization is a systematic and versatile way for doing this. The BV-BRST quantization can be done by the following steps.

The first step is to fermionize the symmetry by introducing anticommuting ghost fields for the infinitesimal parameters of the symmetry. We call the corresponding charge of the fermionic symmetry the BRST charge $\boldsymbol{Q}$. One introduces an additive quantum number U called ghost number or degree and assign $U = 1$ to $\boldsymbol{Q}$. We call the set of original fields and ghosts the set of "fields". One introduces a set of "anti-fields" for the set of "fields" such that in the space $\mathcal{A}$ of all fields (thus "fields" and "anti-fields") one has a natural odd symplectic structure $\boldsymbol{\omega}$. Then we have a corresponding odd Poisson bracket $(.,.)_{BV}$ called the BV bracket among the functions on $\mathcal{A}$. The idea is that one regards "fields" as coordinates, while "anti-fields" are regarded as corresponding conjugate momenta but with the opposite parities (commuting≡even and anti-commuting≡odd) in a certain infinite dimensional phase space. More precisely, one assigns integral ghost number U, or degree, to each field such that $U(\overline{\phi}) = -1 - U(\phi)$, where $\overline{\phi}$ is the "anti-field" of a "field" ϕ. The parity of a field is the same as the parity of its ghost number U. It follows that the odd symplectic form $\boldsymbol{\omega}$ carries $U = -1$, while the BV bracket carries $U = 1$. A BV bracket has the following properties

$$(A, B)_{BV} = -(-1)^{(|A|+1)(|B|+1)}(B, A)_{BV},$$
$$(A, (B, C)_{BV})_{BV} = ((A, B)_{BV}, C)_{BV} + (-1)^{(|A|+1)(|B|+1)}(B, (A, C)_{BV})_{BV},$$
$$(2.1)$$

where A, B, C are (even or odd) local functions on $\mathcal{A}$ and $|A| = U(A)$, etc. We also have the Leibniz law, stating that the BV bracket behaves as a derivation on the ordinary product of functions on $\mathcal{A}$;

$$(A, BC)_{BV} = (A, B)_{BV}C + (-1)^{(|A|+1)|B|}B(A, C)_{BV}. \qquad (2.2)$$

Such a product is (super)-commutative and associative and has degree 0.

One requires, in addition to the odd symplectic structure $\boldsymbol{\omega}$, that $\mathcal{A}$ has a volume element specified by a density ρ compatible with $\boldsymbol{\omega}$. Then one has a BV Laplacian $\triangle_\rho$ defined by $\triangle_\rho A = \frac{1}{2}\text{div}_\rho V_A$, where A is a function on $\mathcal{A}$, V_A is the Hamiltonian vector and div_ρ is the divergence calculated with respect to ρ. The compatibility is the condition $\triangle_\rho^2 = 0$. Note that $\triangle_\rho$ carries ghost number $U = 1$. From now one we shall not specify the density ρ explicitly. In local coordinates $(\{\phi^\alpha\}, \{\overline{\phi}_\alpha\})$ on $\mathcal{A}$ we have $\triangle = \partial^2/\partial\phi^\alpha\partial\overline{\phi}_\alpha$.

Now the BV bracket can be defined by the failure of $\triangle$ being a derivation of the product of functions on $\mathcal{A}$ by the formula

$$(A, B)_{BV} = (-1)^{|A|}\triangle(AB) - (-1)^{|A|}\triangle(A)B + A\triangle(B). \qquad (2.3)$$

The algebra of functions on $\mathcal{A}$ endowed, by the relations (2.1), (2.2) and (2.3), with the bracket $(.,.)_{BV}$ with $U = 1$ generated by $\triangle$ as well as with the (super)-commutative and associative product with $U = 0$, is called a BV algebra.

The BV action functional $\boldsymbol{S}$ is an even function on $\mathcal{A}$ with vanishing ghost number[6], such that (i) its restriction to the subspace of "fields" is the original classical action functional I, (ii) it generates the BRST symmetry via the BV bracket i.e.,

$$(\boldsymbol{S}, \ldots)_{BV} = \boldsymbol{Q}. \qquad (2.4)$$

Equivalently $\boldsymbol{Q}$ is the odd Hamiltonian vector of $\boldsymbol{S}$. (iii) it satisfies the quantum master equation

$$(\boldsymbol{S}, \boldsymbol{S})_{BV} - 2\hbar\triangle\boldsymbol{S} = 0 \equiv \triangle e^{-\frac{\boldsymbol{S}}{\hbar}} = 0, \qquad (2.5)$$

where $\hbar$ denotes Planck constant. A BRST-BV observable $\mathcal{O}$ is a function on $\mathcal{A}$ annihilated by $\boldsymbol{Q} - \hbar\triangle$. The BV master equation is the condition that the expectation value

$$\langle\mathcal{O}\rangle = \int_{\mathcal{L}} d\mu\, \mathcal{O}e^{-\boldsymbol{S}/\hbar} \qquad (2.6)$$

of a BV observable $\mathcal{O}$ is invariant under continuous deformations of the Lagrangian subspace $\mathcal{L}$ with respect to $\boldsymbol{\omega}$ in $\mathcal{A}$. It also implies that the following path integral identically vanishes

$$\langle(-\hbar\triangle + \boldsymbol{Q})A\rangle := \int_{\mathcal{L}} d\mu\, (-\hbar\triangle A + \boldsymbol{Q}A)e^{-\frac{1}{\hbar}\boldsymbol{S}} = 0, \qquad (2.7)$$

for any product of functions A on $\mathcal{A}$. Picking a homology class of a Lagrangian subspace $\mathcal{L}$ of $\mathcal{A}$ is called a gauge fixing.

There are cases in which the classical master equation $(\boldsymbol{S}, \boldsymbol{S}) = 0$ implies the quantum master equation (2.5), i.e., $\triangle\boldsymbol{S} = 0$. Then the master equation implies that $\boldsymbol{S}$ has a fermionic symmetry generated by a nilpotent, $\boldsymbol{Q}^2 = 0$,

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[6]One may allow the action functional to be any even function. Our action functional will always have vanishing ghost number unless specified otherwise.

BRST charge $\boldsymbol{Q}$, which acts on functions on $\mathcal{A}$ as an odd derivation. That is, from (2.1), (2.4), and the classical master equation,

$$\boldsymbol{Q}^2 = 0,$$
$$\boldsymbol{Q}(A, B)_{BV} = (\boldsymbol{Q}A, B)_{BV} + (-1)^{|A|+1}(A, \boldsymbol{Q}B)_{BV}. \tag{2.8}$$

The above structure induces the on the BV algebra a structure of differential BV (dBV) algebra. Remark that $\boldsymbol{Q}$ can be identified with an odd nilpotent vector on $\mathcal{A}$.[7] Now we may further specialize to the case that there are classes of BV observables A_i satisfying $\triangle A_i = 0$. Such observables satisfy $\boldsymbol{Q}A_i = 0$. Note that the BRST charge $\boldsymbol{Q}$ transforms as a scalar under the rotation group of the manifold on which the field theory is defined. In other words the fermionic symmetry is global. Then the BV quantized theory, under the above assumption, is a cohomological field theory, first introduced by Witten. The fixed point theorem of Witten [92] implies that the path integral of $\boldsymbol{Q}$-invariant observables is further localized to an integral over the $\boldsymbol{Q}$-fixed point locus $\mathcal{M}$ in $\mathcal{L}$.

Now we turn to the BV Ward identity. We consider expectation values $\langle A_1 \ldots A_n \rangle$ of products of functions A_i on $\mathcal{A}$. Note that the space $\mathcal{A}$ of all fields is a graded (by the ghost number) superspace. Thus observables also carry ghost numbers. In general any correlation function has vanishing ghost number. Usually the path integral measure carries a ghost number anomaly although the BV action functional has vanishing ghost number. Consequently the net ghost number of $A_1 \ldots A_n$ should cancel the ghost number anomaly in order to have non-vanishing correlation function. Now the identity (2.7) implies that

$$\hbar \langle \triangle(A_1 \ldots A_n) \rangle = \langle \boldsymbol{Q}(A_1 \ldots A_n) \rangle. \tag{2.9}$$

The above identity is called the BV Ward identity, which is non-empty if the net ghost number of $A_1 \ldots A_n$ plus 1 is the total ghost number anomaly. Now consider the case that $\triangle A_i = 0$. From (2.1) and (2.3), we have

$$\hbar \sum_{1 \le j < k \le n} \sigma_{jk} \left\langle (A_j, A_k)_{BV} \prod_{i \neq j,k} A_i \right\rangle = \langle \boldsymbol{Q}(A_1 \ldots A_n) \rangle, \tag{2.10}$$

where σ_{jk} is a sign factor. Now we assume that the classical master equation implies the quantum master equation. Then $\boldsymbol{Q}$ behaves like the exterior

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[7] We will always denote, as is the convention, a derivation with the corresponding vector field by the same symbol.

derivative on field space. Thus the right hand side of above receives contributions only from the boundary in field space. For this one should introduce an appropriate compactification.[8]

2.1.1 Deformations of BV quantized theory

Now we consider deformations of the BV quantized field theory, defined by the triple $(\mathcal{A}, \boldsymbol{\omega}, \boldsymbol{S})$. Note that such a field theory is typically defined on a manifold N and $\mathcal{A}$ is the space of all fields under consideration (the space of all sections of a certain bundle over N), and $\boldsymbol{\omega}$ is an odd symplectic structure on $\mathcal{A}$ with $U = -1$. The action functional $\boldsymbol{S}$ is a function with $U = 0$ defined as an integral of a top-form on N over N, satisfying the BV master equation. For simplicity we assume that $(\boldsymbol{S}, \boldsymbol{S}) = \triangle \boldsymbol{S} = 0$.

Two BV quantized theories $(\mathcal{A}_1, \boldsymbol{\omega}_1, \boldsymbol{S}_1)$ and $(\mathcal{A}_2, \boldsymbol{\omega}_2, \boldsymbol{S}_2)$ are physically equivalent if there is a diffeomorphism $F : \mathcal{A}_1 \to \mathcal{A}_2$ such that $F^*\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1$ and $F^*\boldsymbol{S}_2 = \boldsymbol{S}_1$. It also follows that the odd Hamiltonian vectors $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, respectively, are related as $\boldsymbol{Q}_2 = F_*\boldsymbol{Q}_1$ [29]. Now we consider physically inequivalent deformations of a given theory $(\mathcal{A}, \boldsymbol{\omega}, \boldsymbol{S})$.

We consider a basis $\{\Gamma_a\}$ of $\boldsymbol{Q}$-cohomology among functions on $\mathcal{A}$, which are integrals of top-forms on N over N. Then we can consider a family of action functionals given by $\boldsymbol{S} + t^a\Gamma_a$, where $\{t^\alpha\}$ are formal parameters on a dual basis of $\boldsymbol{Q}$-cohomology. Since $\boldsymbol{Q}\Gamma_a \equiv (\boldsymbol{S}, \Gamma_a) = 0$ the family of action functionals again satisfies the BV master equation up to first order in t^a. To go beyond the first order we should imagine a certain function $\Gamma(t)$, satisfying $\triangle\Gamma(t) = 0$, which has the following formal expansions

$$\Gamma(t) = t^a\Gamma_a + \sum_{n>1} t^{a_1} \ldots t^{a_n}\Gamma_{a_1\ldots a_n}. \tag{2.11}$$

Then we consider the family of action functionals $\boldsymbol{S}(t)$ defined by

$$\boldsymbol{S}(t) = \boldsymbol{S} + \Gamma(t). \tag{2.12}$$

The above deformation is said to be well-defined if $\boldsymbol{S}(t)$ satisfies the BV master equation

$$(\boldsymbol{S}(t), \boldsymbol{S}(t)) = 0 \;\equiv\; \boldsymbol{Q}\Gamma(t) + \frac{1}{2}(\Gamma(t), \Gamma(t)) = 0. \tag{2.13}$$

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[8] For example, all the WDDV type equations (the associativity of quantum cohomology rings, the path integral derivation of Kontsevich's L^∞-quasi-isomorphism, and the A^∞-structure of Fukaya category) are based on elaborations of the above idea.

Obviously $\Gamma_{a_1...a_n}$ for $n > 1$ can not be an element of $\boldsymbol{Q}$-cohomology. We call the deformation (2.12) unobstructed if the solution of $\boldsymbol{Q}\Gamma + \frac{1}{2}(\Gamma, \Gamma) = 0$ has the expansion (2.11). We call the deformation (2.12) classical (or un-extended) if $t^a \neq 0$ only for Γ_a with ghost number $U = 0$, while otherwise $t^a = 0$. We define the extended moduli space $\mathfrak{M}$ of (the well-defined defor-mations of) the theory by the solution space of (2.13) modulo equivalences. We note that the deformed theory (2.12) has a new BRST charge $\boldsymbol{Q}(t)$ defined by $(\boldsymbol{S}(t), \ldots) \equiv \boldsymbol{Q}(t)$. Then the equation (2.13) is equivalent to $\boldsymbol{Q}(t)^2 = 0$ and

$$\boldsymbol{Q}(t)\Gamma(t) = 0. \tag{2.14}$$

We denote call thesubspace $\mathfrak{M}_{cl} \subset \mathfrak{M}$ consisting of solutions with $U = 0$ the classical moduli space. Obviously the tangent space of $\mathfrak{M}$ at a classical point is isomorphic to the $\boldsymbol{Q}$ cohomology group, provided that the deformations are unobstructed.

The above discussion on BV quantization is closely related to modern deformation theory. Details and precise definitions of deformation theory can be found in [50, 7]. Relations with BV quantization are also discussed in [75]. Modern deformation theory associates a certain differential graded Lie algebra (dgLa in short) $\mathfrak{g} = \oplus \mathfrak{g}^k$, modulo quasi-isomorphism, with a mathematical structure being deformed. This gives rise to a formal super-moduli space $\mathfrak{M}_\mathfrak{g}$ defined by the solution space of the Maurer-Cartan (MC in short) equation modulo equivalences,

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0/ \sim \tag{2.15}$$

where $\gamma \in \mathfrak{g}$ and $(d, [.,.])$ denote the differential and the bracket of $\mathfrak{g}$ with degree 1 and 0, respectively;

$$d : \mathfrak{g}^k \rightarrow \mathfrak{g}^{k+1}, \qquad [.,.] : \mathfrak{g}^k \otimes \mathfrak{g}^\ell \rightarrow \mathfrak{g}^{k+\ell}. \tag{2.16}$$

The differential and the bracket have the following properties

$$\begin{aligned}
d^2 &= 0, \\
d[\gamma_1, \gamma_2] &= [d\gamma_1, \gamma_2] + (-1)^{|\gamma_1|}[\gamma_1, d\gamma_2], \\
[\gamma_1, \gamma_2] &= -(-1)^{|\gamma_1||\gamma_2|}[\gamma_2, \gamma_1], \\
[\gamma_1, [\gamma_2, \gamma_3]] &= [[\gamma_1, \gamma_2], \gamma_3] + (-1)^{|\gamma_1||\gamma_2|}[\gamma_2, [\gamma_1, \gamma_3]].
\end{aligned} \tag{2.17}$$

A crucial idea is that one imagines an underlying formal supermanifold $M_\mathfrak{g}$ with a nilpotent odd vector $Q_\mathfrak{g}$ with degree 1 such that d and $[.,.]$ are

the first and second coefficient in the Taylor expansion of $Q_{\mathfrak{g}}$. Then the condition that $Q_{\mathfrak{g}}^2 = 0$ is equivalent to those in (2.17) [2]. Thus

$$Q_{\mathfrak{g}}\gamma = d\gamma + \frac{1}{2}[\gamma, \gamma]. \tag{2.18}$$

Then the moduli space $\mathfrak{M}_{\mathfrak{g}}$ can be identified with the space of $Q_{\mathfrak{g}}$-cohomology $\mathcal{M}_{\mathfrak{g}} = \mathrm{Ker}\, Q_{\mathfrak{g}}/\mathrm{Im}\, Q_{\mathfrak{g}}$, where the quotient by $\mathrm{Im}\, Q_{\mathfrak{g}}$ plays the role of dividing out the equivalence in (2.15). In general, $Q_{\mathfrak{g}}$ may have Taylor components beyond the quadratic terms and those define an L_∞-structure. We may have

$$Q_{\mathfrak{g}}\gamma = d\gamma + \frac{1}{2}[\gamma, \gamma] + \frac{1}{3!}[\gamma, \gamma, \gamma] + \ldots \tag{2.19}$$

A quasi-isomorphism between two dgLas $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is a Q-equivariant map between the two associated formal supermanifolds with base points such that the first Taylor coefficients of the map induces an isomorphism between the cohomologies of $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$. It follows that the $Q_{\mathfrak{g}_1}$ cohomology is isomorphic to that of $Q_{\mathfrak{g}_1}$. Thus the two formal supermoduli spaces $\mathfrak{M}_{\mathfrak{g}_1}$ and $\mathfrak{M}_{\mathfrak{g}_2}$ are isomorphic.

We note that after shifting the degree by -1, the relations in (2.17) are identical to those in (2.1) and (2.8). This implies that one may "represent" a deformation theory by a suitable BV quantized field theory. That is, for a given mathematical structure one wants to deform and the associated dgLa $\mathfrak{g}$, one may consider a BV quantized field theory whose moduli space is isomorphic to the formal super-moduli space $\mathfrak{M}_{\mathfrak{g}}$.[9] Then the quantum field theory gives *additional* information about the deformation theory via correlation functions. What would be the additional information? We do not know the general answer. It may, for example, be possible that deformation problems of different mathematical structures lead to quantum field theories which are physically equivalent, etc. etc. Such phenomena is usually called "dualities" or "mirror symmetry".

2.2 Bosonic string theory as a deformation theory

String theory in the sigma model approach is a theory of maps $u : \Sigma \to X$ from a Riemann surface Σ to a target space X. Picking local coordinates $\{u^I\}$ on X such a map is described by a set of functions $\{u^I(x^\mu)\}$ on Σ,

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[9] We also note that the BV quantization can be generalized to deal with deformation problems involving the full L_∞-algebra [10].

where x^μ, $\mu = 1, 2$, denotes a set of local coordinates on Σ. The bosonic string action in a NS B-field background is given by

$$I' = \frac{1}{2} \int_\Sigma \left(b_{IJ} du^I \wedge du^J \right) \tag{2.20}$$

where b_{IJ} denote the components of the NS 2-form B-field, i.e. $B = \frac{1}{2} b_{IJ} dx^I \wedge dx^J$. Classically a B-field gives a certain structure to the target space X. For instance a non-degenerate B-field corresponds to an almost symplectic structure on X, while a non-degenerate closed (flat $dB = 0$) B-field corresponds to a symplectic structure on X. For our purpose it is convenient to introduce a first order formalism starting from a bivector field $\pi = \frac{1}{2} b^{IJ} \partial_I \wedge \partial_J \in \Gamma(\wedge^2 TX)$, which is not necessarily non-degenerate. We have

$$I = \int_\Sigma \left(H_I \wedge du^I + \frac{1}{2} b^{IJ} H_I \wedge H_J \right), \tag{2.21}$$

where H_I is an "auxiliary" 1-form field in Σ taking values in $u^*(T^*X)$. The boundary condition for $\partial\Sigma \neq 0$ is that H_I vanishes along the tangential direction in $\partial\Sigma$. For a non-degenerate bi-vector, we can integrate out the auxiliary field H_I and we obtain the original action functional (2.20). We note that the action functional I was originally studied for closed Rieman surface in [41, 71]. The authors of [4] also found relation wiht geometric quantization of the symplectic leaves in the target. The path integral approach of Catteo-Felder is based on BV quantization of the above action functional [17]. Here we find it more convenient to make a detour.

The first order formalism above allows us to consider a notion of string theory without an NS background, defined by the following action functional

$$I_o = \int_\Sigma H_I \wedge du^I. \tag{2.22}$$

The resulting theory is obviously topological in two dimensions. It is also invariant under affine transformations of X and does not depends on any other structure in the target space. We may view the string theory in the NS B-field background as a certain deformation of the theory I_o along the "direction" of the bi-vector. We have a criterium for well-defined deformations by requiring that a deformed theory should be a consistent quantum theory. For this we first quantize the theory with action functional I_0 using the BRST-BV formalism. Then we deform the theory along a certain direction and examine if the deformed theory satisfies the quantum master equation.

2.2.1 BV quantization

The quantization of I_0 is rather simple. We note that the action functional I_0 has the following symmetry

$$\delta u^I = 0, \qquad \delta H_I = -d\chi_I, \tag{2.23}$$

where χ_I is an infinitesimal gauge parameter taking values in $u^*(T^*X)$, which vanishes on $\partial\Sigma$ if the boundary of Σ is non-empty. In the BRST quantization one promotes the symmetry to a fermionic one by taking χ_I anticommuting with ghost number $U = 1$. This is equivalent to regarding χ_I as taking value in $u^*(\Pi T^*X)$, where ΠT^*X denotes the parity change of the fiber of T^*X. Now we have the following fermionic symmetry with charge $\boldsymbol{Q}_o$ carrying $U = 1$

$$\boldsymbol{Q}_o u^I = 0, \qquad \boldsymbol{Q}_o H_I = -d\chi_I, \qquad \boldsymbol{Q}_o \chi_I = 0, \tag{2.24}$$

and satisfying $\boldsymbol{Q}_o^2 = 0$. In the present case we introduce a set of "anti-fields" (η_I, ρ^I, v^I) with the ghost numbers $U = (-1, -1, -2)$ for the set of "fields" (u^I, H_I, χ_I). Let $\mathcal{A}$ denote the space of all fields. The "fields" and the "anti-fields" should have a pairing defining a two-form in Σ such that one can integrate over Σ to get a two-form in $\mathcal{A}$. Thus the "anti-fields" (η_I, ρ^I, v^I) are $(2, 1, 2)$-forms on Σ. We can easily find the BV action functional $\boldsymbol{S}_o$ satisfying all the requirements stated in Sect. 2.1 as follows

$$\boldsymbol{S}_o = \int_\Sigma \left(H_I \wedge du^I - \rho^I \wedge d\chi_I \right). \tag{2.25}$$

Thus the BRST transformation laws are

$$\begin{aligned} \boldsymbol{Q}_o u^I &= 0, & \boldsymbol{Q}_o \rho^I &= -du^I, & \boldsymbol{Q}_o v^I &= -d\rho^I, \\ \boldsymbol{Q}_o \chi_I &= 0, & \boldsymbol{Q}_o H_I &= -d\chi_I, & \boldsymbol{Q}_o \eta_I &= -dH_I, \end{aligned} \tag{2.26}$$

satisfying $\boldsymbol{Q}_o^2 = 0$. It is trivial to check the BV master equation

$$(\boldsymbol{S}_o, \boldsymbol{S}_o)_{BV} = 0, \qquad \triangle \boldsymbol{S}_o = 0, \tag{2.27}$$

provided that we have the right boundary conditions mentioned earlier.

A more conceptual and compact formulation can be obtained by combing "fields" and "anti-fields" into two superfields $(\boldsymbol{u}^I, \boldsymbol{\chi}_I)$ carrying $U = (0, 1)$;

$$\begin{aligned} \boldsymbol{u}^I &:= u^I(x^\mu, \theta^\mu) = u^I(x^\mu) + \rho^I_\nu(x^\mu)\theta^\nu - \frac{1}{2}v^I_{\nu_1\nu_2}(x^\mu)\theta^{\nu_1}\theta^{\nu_2}, \\ \boldsymbol{\chi}_I &:= \chi_I(x^\mu, \theta^\mu) = \chi_I(x^\mu) + H_{I\nu}(x^\mu)\theta^\nu + \frac{1}{2}\eta_{I\nu_1\nu_2}(x^\mu)\theta^{\nu_1}\theta^{\nu_2}, \end{aligned} \tag{2.28}$$

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where θ^μ is an anti-commuting vector with $U = 1$. In terms of the superfields we have

$$\boldsymbol{S}_o = \int d^2\theta \; \boldsymbol{\chi}_I D\boldsymbol{u}^I, \tag{2.29}$$

and

$$\boldsymbol{Q}_o \boldsymbol{u}^I = D\boldsymbol{u}^I, \qquad \boldsymbol{Q}_o \boldsymbol{\chi}_I = D\boldsymbol{\chi}_I, \tag{2.30}$$

where $D = \theta^\mu \partial_\mu$. The natural odd symplectic form $\boldsymbol{\omega}$ in $\mathcal{A}$ is defined by

$$\boldsymbol{\omega} = \int_\Sigma d^2\theta \left(\boldsymbol{\delta u}^I \boldsymbol{\delta\chi}_I \right), \tag{2.31}$$

where $\boldsymbol{\delta}$ denotes the exterior differential on $\mathcal{A}$. The odd symplectic form $\boldsymbol{\omega}$ has degree $U = -1$ since $d^2\theta$ shift the degree by $U = -2$. The BV bracket is the odd Poisson bracket among functions in $\mathcal{A}$ with respect to $\boldsymbol{\omega}$. The BRST charge $\boldsymbol{Q}_o$ can be identified with the odd Hamiltonian vector of $\boldsymbol{S}_o$;

$$\boldsymbol{i}_{\boldsymbol{Q}_o}\boldsymbol{\omega} = \boldsymbol{\delta S}_o, \; equivalently \; (\boldsymbol{S}_o, \ldots)_{BV} = \boldsymbol{Q}_o. \tag{2.32}$$

Now we observe that the superfields $(\boldsymbol{u}^I, \boldsymbol{\chi}_I)$ parametrize maps

$$\boldsymbol{\phi} : \Pi T\Sigma \to \Pi T^*X \tag{2.33}$$

between the two superspaces. Here $\Pi T\Sigma$ denotes the total space of the tangent bundle of Σ after parity change of the fiber. We regard $(\{x^\mu\}, \{\theta^\mu\})$ as a set of local coordinates on $\Pi T\Sigma$. With the target superspace ΠT^*X we mean the total space of the cotangent bundle of X after parity change of the fiber. We denote a set of local coordinates on ΠT^*X by $(\{u^I\}, \{\chi_I\})$ carrying the ghost number $U = (\{0\}, \{1\})$.[10] Thus the space $\mathcal{A}$ of all fields is the space of all maps above. The odd symplectic form $\boldsymbol{\omega}$ (2.31) on $\mathcal{A}$ is the unique extension of the odd symplectic form $\omega = du^I d\chi_I$ with degree $U = 1$ on ΠT^*X to $\mathcal{A}$. More precisely

$$\boldsymbol{\omega} \equiv \int_\Sigma d^2\theta \; \phi^*(\omega), \tag{2.34}$$

where ϕ^* is the pull-back of a map $\boldsymbol{\phi} : \Pi T\Sigma \to \Pi T^*X$. The odd Poisson bracket $[.,.]_S$ with degree $U = -1$ among functions γ_i on ΠT^*X is called the Schouten-Nijenhuis bracket;

$$[\gamma_1, \gamma_2]_S := \frac{\partial\gamma_1}{\partial\chi_I}\frac{\partial\gamma_2}{\partial u^I} - (-1)^{(|\gamma_1|-1)(|\gamma_2|-1)}\frac{\partial\gamma_2}{\partial\chi_I}\frac{\partial\gamma_1}{\partial u^I} \tag{2.35}$$

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[10]By abuse of notation, u^I may denote both a coordinate on X or a function $u^I(x^\mu)$ on Σ etc. This should not cause any confusion in the present context.

where $|\gamma|$ denotes the degree of γ. We remark that a function on $\Pi T^* X$ with $U = p$ is a p-vector (an element of $\Gamma(X, \wedge^p T X)$) after parity change. We also remark that functions on $\Pi T^* X$ after shifting the degree by 1 together with the Schouten-Nijenhuis bracket form a dgLa with zero differential. The product of functions on $\Pi T^* X$ corresponds to wedge products of multivectors on X. We also note that a function γ on $\Pi T^* X$ induces a function $\int_\Sigma d^2\theta \, \phi^*(\gamma)$ on $\mathcal{A}$ with $U = |\gamma| - 2$. It follows that

$$\left(\int_\Sigma d^2\theta \, \phi^*(\gamma), \int_\Sigma d^2\theta \, \phi^*(\gamma) \right)_{BV} = \int_\Sigma d^2\theta \, \phi^*([\gamma, \gamma]_S). \qquad (2.36)$$

2.2.2 Deformations

Now we examine deformations of the theory. A 0-dimensional observable with ghost number $U = p$ may be any function on $\mathcal{A}$ of the form

$$\gamma^{(0)} = \frac{1}{p!} \gamma^{I_1 \dots I_p}(u^I(x^\mu)) \chi_{I_1}(x^\mu) \dots \chi_{I_p}(x^\mu). \qquad (2.37)$$

Any such $\gamma^{(0)}$ satisfies $\boldsymbol{Q}_o \gamma^{(0)} = 0$ since $\boldsymbol{Q}_o u^I = \boldsymbol{Q}_o \chi_I = 0$. It is obvious that no such $\gamma^{(0)}$ can be $\boldsymbol{Q}_o$-exact. Thus an arbitrary functional $\gamma(u^I(x^\mu), \chi_J(x^\mu))$ belongs to the $\boldsymbol{Q}_o$ cohomology group. It is also obvious that $\triangle \gamma^{(0)} = 0$. The coefficients $\gamma^{I_1 \dots I_p}$ can be identified with coefficients of a multivector $\Gamma(X, \wedge^p T X)$. Thus the space of 0-dimensional observables is isomorphic to the space of all multivectors on X. Now we denote by $\boldsymbol{\gamma} := \gamma^{(0)}(\boldsymbol{u}^I, \boldsymbol{\chi}_I)$ the corresponding function of superfields $(\boldsymbol{u}^I, \boldsymbol{\chi}_I)$. Equivalently $\boldsymbol{\gamma} := \phi^*(\gamma)$. Then we have the following expansions

$$\boldsymbol{\gamma} = \gamma^{(0)} + \left(\mathcal{K}_\mu \gamma^{(0)} \right) \theta^\mu + \frac{1}{2} \left(\mathcal{K}_\mu \mathcal{K}_\nu \gamma^{(0)} \right) \theta^\mu \theta^\nu, \qquad (2.38)$$

where $\mathcal{K}_\mu := -\frac{\partial}{\partial \theta^\mu}$. We note that $\mathcal{K}_\mu$ is anticommuting and carries $U = -1$. Together with $\boldsymbol{Q}_o$ we have the following anticommutation relations

$$\{\boldsymbol{Q}_o, \boldsymbol{Q}_o\} = 0, \qquad \{\boldsymbol{Q}_o, \mathcal{K}_\mu\} = -\partial_\mu, \qquad \{\mathcal{K}_\mu, \mathcal{K}_\nu\} = 0. \qquad (2.39)$$

The above relations and (2.38) imply that $\gamma^{(n)} := \frac{1}{n!} \left(\mathcal{K}_{\mu_1} \dots \mathcal{K}_{\mu_n} \gamma \right) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$, where $n = 0, 1, 2$, satisfy the descent equations

$$\begin{aligned} \boldsymbol{Q}_o \gamma^{(0)} &= 0, \\ \boldsymbol{Q}_o \gamma^{(1)} &= -d\gamma^{(0)}, \\ \boldsymbol{Q}_o \gamma^{(2)} &= -d\gamma^{(1)}. \end{aligned} \qquad (2.40)$$

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Consequently $\int_\Sigma \gamma^{(2)} \equiv \int_\Sigma d^2\theta \, \boldsymbol{\gamma}$ is invariant under $\boldsymbol{Q}_o$, if $\gamma^{(1)} = 0$ at the boundary $\partial\Sigma$ of Σ, i.e., $\boldsymbol{Q}_o \int_\Sigma \gamma^{(2)} = -\int_{\partial\Sigma} \gamma^{(1)}$. It is not difficult to check $\triangle \int_\Sigma d^2\theta \, \boldsymbol{\gamma} = 0$, with suitable regularization [17].

Provided that $\gamma^{(1)} = 0$ at the boundary $\partial\Sigma$, $\int_\Sigma d^2\theta \, \boldsymbol{\gamma}$ is a 2-dimensional observable, which can be used to deform the theory;

$$\boldsymbol{S}_\gamma = \boldsymbol{S}_o + \int_\Sigma d^2\theta \, \boldsymbol{\gamma}. \tag{2.41}$$

The above action functional satisfies $\boldsymbol{Q}_o \boldsymbol{S}_\gamma = \triangle \boldsymbol{S}_\gamma = \mathcal{K}_\mu \boldsymbol{S}_\gamma = 0$. It follows, from the relation (2.36), that the deformed action functional satisfies the BV master equation if $[\gamma, \gamma]_S = 0$. Note that the ghost number of $\int_\Sigma d^2\theta \, \boldsymbol{\gamma}$ is shifted by 2 from the ghost number of $\boldsymbol{\gamma}$. We can view $\boldsymbol{\gamma}$ as the pull-back of a function on $\Pi T^* X$ by a map $\Pi T\Sigma \to \Pi T^* X$. Thus an even (odd) function on $\Pi T^* X$ leads to an even (odd) function on $\mathcal{A}$.

Now the boundary condition introduced earlier implies that $\gamma^{(1)} = 0$ at the boundary iff $\gamma(u^I, \chi_I)|_{\chi_I=0} = 0$ for all I. Equivalently $\gamma|_X = 0$. Then we may use $\int_\Sigma d^2\theta \, \boldsymbol{\gamma}$ to deform the theory. On the other hand such a function γ on $\Pi T^* X$ does not give non-trivial observables supported on the boundary. Non-trivial boundary observables must come from functions on the base space X of $\Pi T^* X$, which can not be used to deform the bulk theory.

The above considerations lead us to consider the following action functional with $U = 0$

$$\boldsymbol{S} = \boldsymbol{S}_o + \int_\Sigma d^2\theta \, \boldsymbol{\pi}, \tag{2.42}$$

where $\boldsymbol{\pi} = \frac{1}{2} b^{IJ}(\boldsymbol{u}^K) \boldsymbol{\chi}_I \boldsymbol{\chi}_J$ satisfying

$$\left(\int_\Sigma d^2\theta \, \boldsymbol{\pi}, \int_\Sigma d^2\theta \, \boldsymbol{\pi} \right)_{BV} = 0. \tag{2.43}$$

The above condition implies that the action functional $\boldsymbol{S}$ satisfies the master equation $(\boldsymbol{S}, \boldsymbol{S})_{BV} = 0$. A parity changed bivector $\pi = \frac{1}{2} b^{IJ}(u^K) \chi_I \chi_J$ on X satisfying $[\pi, \pi]_S = 0$ is called a Poisson bi-vector. Since $\boldsymbol{S}$ is an even function on the space of all fields (the space of all maps $\Pi T\Sigma \to \Pi T^* X$) $\mathcal{A}$ we have an odd Hamiltonian vector $\boldsymbol{Q}$ defined by $\boldsymbol{i_Q \omega} = \boldsymbol{\delta S}$ or, equivalently $(\boldsymbol{S}, \ldots)_{BV} = \boldsymbol{Q}$, regarding the Hamiltonian vector $\boldsymbol{Q}$ as an odd derivation. Thus $\boldsymbol{Q}$ is the BV-BRST charge of the action functional $\boldsymbol{S}$. Explicitly

$$\boldsymbol{Q} = \left(\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\chi}_I} + D\boldsymbol{u}^I \right) \frac{\partial}{\partial \boldsymbol{u}^I} + \left(\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{u}^I} + D\boldsymbol{\chi}_I \right) \frac{\partial}{\partial \boldsymbol{\chi}_I}. \tag{2.44}$$

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The BV master equation is equivalent to the condition that $\boldsymbol{Q}$ is nilpotent. Finally we note that

$$\{\boldsymbol{Q}, \boldsymbol{Q}\} = 0, \qquad \{\boldsymbol{Q}, \mathcal{K}_\mu\} = -\partial_\mu, \qquad \{\mathcal{K}_\mu, \mathcal{K}_\nu\} = 0. \qquad (2.45)$$

The bosonic part of the action functional $\boldsymbol{S}$ is given by the classical action functional I in (2.21). By construction, compare with [17], $\boldsymbol{S}$ is obtained by BV quantization of I. We also mention that the anti-commutation relation (2.45) should be compared with the relations $L_{-1} = \{Q, b_{-1}\}$ and $\overline{L}_{-1} = \{Q, \overline{b}_{-1}\}$ in 2-dimensional conformal field theory (see in particular [100] on closely related issues with this paper) or with a twisted $N_{ws} = (2, 2)$ world-sheet supersymmetry [89].

2.3 Open string and formality

Now we recall the solution of deformation quantization based on the path integral approach.

We first recall some basic properties of associative algebras (see [50, 83]). On any associative algebra A we have the Hochschild co-chain complex $(\delta, \oplus_n C^n(A, A))$ where $C^n(A, A)$ is the space of linear maps $A^{\otimes n} \to A$ and $\delta : C^\bullet(A, A) \to C^{\bullet+1}(A, A)$ with $\delta^2 = 0$ is the Hochschild differential. For $A = C^\infty(X)$ the Hochschild co-chains are identified with the space of multidifferential operators. It is known that the Hochschild cohomology $HH^\bullet(A, A)$ is isomorphic to the space of multivectors $\Gamma(X, \wedge^\bullet TX)$.[11] We recall that the space $\Gamma(X, \wedge^{\bullet+1} TX) \equiv HH^{\bullet+1}(A, A)$ together with the Schouten-Nijenhuis bracket $[.,.]_S$ and zero-differential form forms the dgLa called $T_{poly}^\bullet(X)$. This dgLa originates from another dgLa called $D_{poly}(X)$ on the space $C^{\bullet+1}(A, A)$ with the Gerstenhaber (G in short) bracket $[.,.]_G$ and differential δ and associative cup product. The G bracket

$$[.,.]_G : C^n(A, A) \otimes C^m(A, A) \to C^{m+n-1}(A, A) \qquad (2.46)$$

is defined as follows. For $\Phi_1 \in C^n(A, A)$ and $\Phi_2 \in C^m(A, A)$

$$[\Phi_1, \Phi_2]_G = \Phi_1 \circ \Phi_2 - (-1)^{(n-1)(m-1)} \Phi_2 \circ \Phi_1, \qquad (2.47)$$

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[11]Thus the $\boldsymbol{Q}_o$-cohomology is isomorphic to the total Hochschild cohomology $\oplus_n HH^n(A, A)$.

where

$$\Phi_1 \circ \Phi_2(a_1, \ldots, a_{n+m-1})$$
$$= \sum_{j=0}^{n-1} (-1)^{(m-1)j} \Phi_1(a_1, \ldots, a_j, \Phi_2(a_{j+1}, \ldots, a_{j+m}), \ldots). \qquad (2.48)$$

The differential $\delta : C^\bullet(A, A) \to C^{\bullet+1}(A, A)$ is defined by $\delta = [m, \ldots]_G$ where $m(a, b) = ab$ for $a, b \in A$. Then $[m, m]_G = 0$ is equivalent to m being associative, which implies that $\delta^2 = 0$.

Now for a given manifold X a star product $f \star g$ among two functions f and g is defined by

$$f \star g = fg + \Pi(f, g), \qquad (2.49)$$

where $\Pi \in D^1_{poly}(X)$ has a formal expansion

$$\Pi(f, g) = \Pi_1(f, g)\hbar + \Pi_2(f, g)\hbar^2 + \ldots. \qquad (2.50)$$

The associativity of the star product can be written in terms of the MC equation for $D^1_{poly}(X)$ as

$$\delta\Pi + \frac{1}{2}[\Pi, \Pi]_G = 0 \equiv [m + \Pi, m + \Pi]_G = 0. \qquad (2.51)$$

To first approximation the associativity implies that $\Pi(f, g)_1 = \langle \pi, df \otimes dg \rangle \equiv \frac{1}{2}\{f, g\}$ and the bracket $\{., .\}$ satisfies the Jacobi identity. Namely the bivector π is Poisson and $\{., .\}$ is the Poisson bracket. Equivalently

$$[\pi, \pi]_S = 0, \qquad (2.52)$$

which is the MC equation for $T^1_{poly}(X)$.

Consequently the problem of deformation quantization for an arbitrary Poisson manifold X is equivalent to proving isomorphism between two moduli spaces defined by the two MC equations (2.51) and (2.52). Kontsevich proved the more general theorem (L_∞ formality) that the dgLa of multivectors on X with vanishing differential and Schouten-Nijenhuis bracket is quasi-isomorphic to the dgla of multi-differential operators on X. He also gave an explicit expression for his quasi-isomorphism in the case of $X = \mathbb{R}^n$, essentially summing over contributions from certain graphs resembling Feynman path integral.

Catteneo-Felder studied the path integral of the theory defined by the action functional $\boldsymbol{S}$ on the disk $\Sigma = D$ with boundary punctures, where observables constructed from functions on X are inserted. They obtained

Kontsevich's explicit formula by perturbative expansions around constant maps [17]. This formulation essentially maps the space of polyvectors on the target space X with vanishing Schouten-Nijenhuis bracket modulo equivalence to the space of solution of the BV master equation of the theory deformed along the direction of polyvectors modulo equivalences. The BV master equation then implies according to (2.10) that the perturbative expansion of the theory defines a quasi-isomorphism. That is, the MC equation (2.52) for $T^1_{poly}(X)$ is the BV master equation, while the MC equation (2.51) is the Ward identity.

We remark that an associative algebra A is a 1-algebra and its Hochschild cochain complex $(C^\bullet(A, A), \delta)$ has the structure of a 2-algebra by the cup product and the G-bracket $[., .]_G$. We note that the cup product is (super)-commutative only up to homotopy. The cohomology of the Hochschild complex is isomorphic to the space of polyvectors on X and also has a structure of a 2-algebra with the Schouten-Nijenhuis bracket (induced from G-bracket) and the wedge product (induced from the cup product), which is (super)-commutative. We call the latter the cohomological 2-algebra. The formality theorem of Kontsevich means that the 2-algebra and its cohomology are equivalent up to homotopy as a L_∞ algebra, thus forgetting the product structure, for $A = C(X)$. Tarmakin generalized the formality to the category of G_∞ algebras for $X = \mathbb{R}^n$ [79]. Note that the Schouten-Nijenhuis bracket can be generated by $\triangle = \frac{\partial^2}{\partial u^i \partial \chi_i}$ for $X = \mathbb{R}^n$. Thus the cohomological 2-algebra is a BV algebra. The BV_∞-formality is conjectured in [55]. We do not want to review here all those purely algebraic approaches (see however [51]). We should also mention that those are closely related with the open-closed string field theory of Zwiebach. For us it should suffice to meditate OS.

2.4 A model

In this subsection we reexamine the A model of the topological sigma model. The A model was originally defined for an arbitrary almost complex manifold [89]. Our presentation for the A model will be similar to that of the Catteneo-Felder model in the previous section. Here, however, we will take the opposite direction by starting from a dGBV algebra associated with a symplectic manifold X. Then we will construct the corresponding two-dimensional sigma model, which leads to the A model after gauge fixing. A similar approach to the A model in the Kähler case is discussed in [2]. We recall that the A model for the Kähler case is a twisted version of a

$N_{ws} = (2, 2)$ world-sheet supersymmetric sigma model.

2.4.1 Covariant Schouten-Nijenhuis bracket

Consider a manifold X with a Poisson bi-vector π, which corresponds to even function $\pi = \frac{1}{2} b^{IJ} \chi_I \chi_J$ on $\Pi T^* X$ with $[\pi, \pi]_S = 0$. The associated odd nilpotent Hamiltonian vector Q_π is given by

$$Q_\pi = b^{IJ} \chi_J \frac{\partial}{\partial u^I} + \frac{1}{2} \partial_K b^{IJ} \chi_I \chi_J \frac{\partial}{\partial \chi_K}. \tag{2.53}$$

Since $Q_\pi^2 = 0$, Q_π defines a cohomology on $\Pi\Gamma(\wedge^\bullet TX)$. The resulting cohomology is called the Poisson cohomology $H_\pi^\bullet(X)$ of X [59]. For the symplectic case we are considering the Poisson cohomology is isomorphic to de Rham cohomology.

Now we consider the dual picture in ΠTX. We introduce natural local coordinates (u^I, ψ^I) on ΠTX. For any differentiable manifold X we have a distinguished odd vector Q in ΠTX

$$Q = \psi^I \frac{\partial}{\partial u^I} \tag{2.54}$$

of degree $U = 1$ with $Q^2 = 0$. It is obtained by the parity change of the exterior derivative d on X. Any differential form on X corresponds to a function on ΠTX. The Q-cohomology is isomorphic to the de Rham cohomology. We consider a Poisson bi-vector π on X and define the associated contraction operator i_π;

$$i_\pi = \frac{1}{2} b^{IJ} \frac{\partial^2}{\partial \psi^I \partial \psi^J}. \tag{2.55}$$

Then we define an odd second order differential operator $\triangle$ with degree $U = -1$ by

$$\triangle_\pi := [i_\pi, Q] = b^{IJ} \frac{\partial^2}{\partial u^I \partial \psi^J} + \frac{1}{2} \partial_I b^{JK} \psi^I \frac{\partial^2}{\partial \psi^I \partial \psi^J}. \tag{2.56}$$

It is not difficult to show that the condition $[\pi, \pi]_S = 0$ implies

$$\triangle_\pi^2 = [Q, \triangle_\pi] = 0. \tag{2.57}$$

The operator $\triangle_\pi$ acting on polynomial functions in ΠTX is the parity changed version of the Koszul-Brylinski boundary operator, which defines

the Poisson homology $H_\bullet^\pi(X)$ [54, 15]. If X is an unimodular Poisson manifold (including the symplectic case) with dimensions n we have the duality $H_p^\pi(X) = H_\pi^{n-p}(X)$. It follows that an element $\alpha \in H^\bullet(X)$ satisfies $Q\alpha = \triangle_\pi \alpha = 0$ for a symplectic manifold X. For a Kähler manifold we have $\triangle_\pi = i\partial^* - i\overline{\partial}^*$ by the Kähler identity. From now on we omit the subscript π from $\triangle_\pi$.

The operator $\triangle$ allows us to define an odd Poisson structure on ΠTX, which is the covariant version of the Schouten-Nijenhuis bracket [54]. Let a, b denote functions on ΠTX with degree $U = |a|, |b|$. The ordinary product $a \cdot b$ originates from the wedge product on $\Gamma(\wedge^\bullet T^*X)$. It follows that Q is a derivation while $\triangle$ fails to be a derivation. One defines the covariant Schouten-Nijenhuis bracket by the formula

$$[a \bullet b] = (-1)^{|a|}\triangle(a \cdot b) - (-1)^{|a|}\triangle a \cdot b - a \cdot \triangle b. \qquad (2.58)$$

It is not difficult to check that $(Q, \triangle, \cdot, \mathcal{O}(\Pi TX)$ form a dGBV algebra [62].

From now on we assume that X is a symplectic manifold. A symplectic manifold is a Poisson manifold with a non-degenerated Poisson bi-vector π Let b_{IJ} be the inverse of b^{IJ}. Then the condition $[\pi, \pi]_S = 0$ is equivalent to $d\omega = 0$, where $\omega = \frac{1}{2}b_{IJ}du^I \wedge du^J$. Let $B = \frac{1}{2}b_{IJ}\psi^I\psi^J$ be the parity changed symplectic form. It is not difficult to show that

$$\triangle B = 0, \qquad [B \bullet B] = 0, \qquad [B \bullet a] = Qa. \qquad (2.59)$$

Thus Q is the Hamiltonian vector of B.

2.4.2　A BV sigma model

Now we consider a two dimensional sigma model which is a theory of maps $\Pi T\Sigma \to \Pi TX$. We denote local coordinates in $\Pi T\Sigma$ by (x^μ, θ^μ), $\mu = 1, 2$. We denote $(\boldsymbol{u}^I := u^I(x^\mu, \theta^\mu), \boldsymbol{\psi}^I := \psi^I(x^\mu, \theta^\mu)$ as the extension of (u^I, χ^I) to functions on $\Pi T\Sigma$. Let $\mathcal{A}$ denote the space of all maps $\phi : \Pi T\Sigma \to \Pi TX$. The odd vector Q on ΠTX can naturally be extended to an odd vector $\boldsymbol{Q}$ on $\mathcal{A}$;

$$\boldsymbol{Q} = \boldsymbol{\psi}^I \frac{\partial}{\partial \boldsymbol{u}^I}. \qquad (2.60)$$

It is obvious that $\boldsymbol{Q}^2 = 0$. We define the action functional $\boldsymbol{S}$ of the theory by $\int_\Sigma d^2\theta \, \phi^*(B)$;

$$\boldsymbol{S} = \frac{1}{2} \int d^2\theta \, \left(\boldsymbol{b}_{IJ}\boldsymbol{\psi}^I\boldsymbol{\psi}^J\right). \qquad (2.61)$$

The odd symplectic structure on ΠTX induces an odd Poisson structure on $\mathcal{A}$. The BV operator $\triangle$ is defined by the formula $\triangle := [i_{\boldsymbol{\pi}}, \boldsymbol{Q}]$. The associated BV bracket, denoted by $(.,.)_{BV}$, is the BV bracket of the theory. It follows from (2.59) that

$$\triangle \boldsymbol{S} = 0, \qquad (\boldsymbol{S}, \boldsymbol{S})_{BV} = 0, \qquad (\boldsymbol{S}, \ldots)_{BV} = \boldsymbol{Q}. \qquad (2.62)$$

Thus the action functional satisfies the quantum master equation.

We may expand the two superfields $(\boldsymbol{u}^I, \boldsymbol{\psi}^I)$ carrying $U = (0,1)$; as follows

$$\boldsymbol{u}^I = u^I + \rho_\mu^I \theta^\mu + \frac{1}{2} v_{\mu\nu}^I \theta^\mu \theta^\nu,$$
$$\boldsymbol{\psi}^I = \psi^I + H_\mu^I \theta^\mu + \frac{1}{2} \eta^I{}_{\mu\nu} \theta^\mu \theta^\nu. \qquad (2.63)$$

As before the ghost number of θ^μ is assigned the ghost number $U = 1$. The $\boldsymbol{Q}$ transformation law is

$$\boldsymbol{Q} u^I = \psi^I, \qquad \boldsymbol{Q} \psi^I = 0. \qquad (2.64)$$

The explicit form of the action functional $\boldsymbol{S}$ is

$$\boldsymbol{S} = \int_\Sigma d\theta^2 \left(\frac{1}{2} b_{IJ} H^I \wedge H^J + 2 b_{IJ} \eta^I \psi^J + \partial_K b_{IJ} \rho^K \wedge H^I \psi^J \right.$$
$$\left. + \frac{1}{2} \partial_K \partial_L b_{IJ} \rho^K \wedge \rho^L \psi^I \psi^J + \partial_K b_{IJ} v^K \psi^I \psi^J \right). \qquad (2.65)$$

2.4.3 Gauge fixing

Recall that the quantum master equation is the condition that the path integral of the theory $\langle \mathcal{O} \rangle = \int_{\mathcal{L}} d\mu \, \mathcal{O} e^{-\frac{1}{\hbar} \boldsymbol{S}}$ restricted to a Lagrangian submanifold $\mathcal{L} \subset \mathcal{A}$ is invariant under smooth deformations of $\mathcal{L}$, provided that $\mathcal{O}$ is a BV observable, $(\boldsymbol{Q} - \hbar \triangle) \mathcal{O} = 0$. Picking a homology class of $\mathcal{L}$ is called BV gauge fixing. In the present case all the observables $\mathcal{O}_\alpha$ will satisfy $\triangle \mathcal{O}_\alpha = 0$. Thus $\mathcal{O}_\alpha$ should be invariant under $\boldsymbol{Q}$. Note also that the action functional $\boldsymbol{S}$ is also invariant under $\boldsymbol{Q}$. Then we can apply the fixed point theorem of Witten [92], since $\boldsymbol{Q}$ generates a global fermionic symmetry on Σ. According to this theorem the path integral is localized to an integral over the fixed point locus of $\boldsymbol{Q}$. Consequently the path integral is localized to an integral over the fixed point locus $\mathcal{L}_0$ in the Lagrangian subspace $\mathcal{L}$. Note that the space of all fields $\mathcal{A}$ and its Lagrangian subspace $\mathcal{L}$ are both infinite dimensional superspaces. Thus the path integral is difficult to make

sense of. However we can reduce the integral to an finite dimensional subspace $\mathcal{L}_0$ due to the fixed point theorem. Now we look for an appropriate gauge fixing.

On any symplectic manifold (X, ω) there is an almost complex structure $J \in \Gamma(End(TX))$ compatible with B. The almost complex structure J^I_J obeying

$$J^I{}_K J^K{}_J = -\delta^I{}_K \tag{2.66}$$

is compatible with b_{IJ} in the sense that

$$b_{IJ} = J^K{}_I J^L{}_J b_{KL}. \tag{2.67}$$

Thus b^{IJ} is of type $(1,1)$. The above relation is equivalent to the following condition

$$g_{IJ} = g_{JI}, \tag{2.68}$$

where $g_{IJ} = b_{IK} J^K{}_J$ is a Riemannian metric with torsion free connection. We want to gauge fix the theory such that the path integral is localized to the moduli space of pseudo-holomorphic maps

$$du^I + J^I{}_K * du^K = 0, \tag{2.69}$$

where d and $*$, $*^2 = -1$ denote the exterior derivative and the Hodge star in Σ. Note that the transformation laws of the anti-ghost multiplet (ρ^I, H^I) are

$$\boldsymbol{Q}\rho^I = -H^I, \qquad \boldsymbol{Q}H^I = 0. \tag{2.70}$$

Thus $H^I = 0$ at the fixed point locus. The appropriate gauge fixing should be $H^I = du^I + J^I{}_K * du^K = 0$. For this purpose we should impose the "self-duality" condition of Witten [89]

$$\begin{aligned}
\rho^I &= *J^I{}_K \rho^K, \\
H^I &= *J^I{}_K H^K.
\end{aligned} \tag{2.71}$$

Then, as explained by Witten, the $\boldsymbol{Q}$ transformation law on the "self-dual" part ρ^{+I} of ρ^I should be

$$\boldsymbol{Q}\rho^{+I} = -H^{+I} - \frac{1}{2}\left(\mathcal{D}_K J^I{}_J\right)\psi^K * \rho^{+J} + \Gamma^I{}_{JK}\psi^J \rho^{+K}, \tag{2.72}$$

where $\mathcal{D}_K$ denotes the covariant derivative. In the above the term proportional to $\mathcal{D}^K J^I{}_L$ is needed for the compatibility of $\boldsymbol{Q}$ with the "self-duality" (2.71).

$$\boldsymbol{Q}\rho^{+I} = *\left(\boldsymbol{Q}J^I{}_K\right)\rho^{+K} + J^I{}_K \boldsymbol{Q}\rho^{+K}. \tag{2.73}$$

The last term is needed for covariance with respect to reparameterizations of u^I. For this we regard (H_I^-, ρ^{+I}) as fields, while (ρ^{-I}, H_I^+) are the corresponding anti-fields. Now we pick the following gauge fermion Ψ with $U = -1$

$$\Psi = \int_\Sigma b_{IJ}\rho^{+I} \wedge \left(du^J + \frac{1}{2} \left(\mathcal{D}_K J^J{}_L \right) \psi^K * \rho^{+L} - \Gamma^J{}_{KL}\psi^K \rho^{+L} \right). \quad (2.74)$$

Now the Lagrangian submanifold $\mathcal{L}$ is determined by the following equations;

$$b_{IJ}v^J = \frac{\delta\Psi}{\delta\psi^I}, \quad b_{IJ}\rho^{-I} := \frac{\delta\Psi}{\delta H^{-I}}, \quad b_{IJ}H^{+I} := \frac{\delta\Psi}{\delta\rho^{+I}}, \quad b_{IJ}\eta^J := \frac{\delta\Psi}{\delta u^I}. \quad (2.75)$$

Thus we have, for instance,

$$\rho^{-I} = 0,$$
$$H^I = \left(du^I + *J^I{}_K du^K \right) + \frac{1}{2} \left(\mathcal{D}_K J^I{}_L \right) \psi^K * \rho^{+L} - \Gamma^I{}_{KL}\psi^K \rho^{+L}, \quad (2.76)$$

as desired. Then the gauge fixed action functional (or the action restricted as a function on $\mathcal{L}$) is exactly the action functional of Witten's topological sigma model [89]. The transformation laws of ρ^I after the gauge fixing (restricted to $\mathcal{L}$) is then

$$\boldsymbol{Q}\rho^I = - \left(du^I + *J^I{}_K du^K \right) - \frac{1}{2} \left(\mathcal{D}_K J^I{}_J \right) \psi^K * \rho^J + \Gamma^I{}_{JK}\psi^J \rho^K. \quad (2.77)$$

Since $\psi^I = 0$ at the fixed point the fixed point locus $\mathcal{L}_0$ in $\mathcal{L}$ is the space of pseudo holomorphic maps.

3 Topological Open Membrane

We want to study the theory of membranes in the background of a C-field but no metric. The theory is defined by maps from a 3-dimensional world-volume N with boundary to the target space X. We assume that the canonical line bundle $\det(T^*N)$ is trivial so that we have a well-defined volume element. We describe a map $u : N \rightarrow X$ locally by functions $u^I(x^\mu)$ where $\{x^\mu\}$, $\mu = 1, 2, 3$ are local coordinates on N. We consider a 3-form $C = \frac{1}{3!}c_{IJK}(u^L)du^I \wedge du^J \wedge du^K$ in X, where $\{u^I\}$ denote local coordinates

on X. Now the action functional of the membrane coupled to a C-field is, for $\partial N = 0$,

$$I_1' = \int_N u^*(C) = \frac{1}{3!} \int_N c_{IJK}(u^L) du^I \wedge du^J \wedge du^K. \qquad (3.1)$$

To adopt the same strategy as for the string theory, we rewrite the action in the following first order formalism:

$$I_1 = \int_N \left(F_I^{(2)} \wedge \left(A^I - du^I \right) - A^I \wedge dH_I + \frac{1}{3!} c_{IJK}(u^L) A^I \wedge A^J \wedge A^K \right), \qquad (3.2)$$

where A^I are 1-forms in N taking values in $u^*(TX)$, while $F_I^{(2)}$ and H_I are 2 and 1-forms, respectively, taking values in $u^*(T^*X)$. The algebraic equation of motion of $F_I^{(2)}$ imposes the constraint $A^I - du^I = 0$ and the H_I equation of motion $dA^I = 0$ gives the integrability condition. Thus we have a well-defined first order formalism. On shell we recover the action functional I_1' in (3.1). We might try to do BV quantization of the first order action functional I_1. It is however more appropriate to consider the membrane theory without background and study consistent deformations.

We may define the bosonic membrane theory without background by the following action functional

$$I_o = \int_N \left(F_I^{(2)} \wedge \left(A^I - du^I \right) - A^I \wedge dH_I \right), \qquad (3.3)$$

where all fields above carry ghost number $U = 0$. From now on we remove the restriction $\partial N = 0$. The boundary conditions are such that $A^I(x)$ and $*F_I^{(2)}$ vanish along the directions tangent to ∂N while $H_I(x)$ vanishes along the direction normal to ∂N for $x \in \partial N$. On shell we have the boundary string without background,

$$I_o|_{\text{on shell}} = \int_{\partial N} H_I \wedge du^I, \qquad (3.4)$$

as a bonus of the integrability of the first order formalism. We note that I_o is invariant, up to a total derivative, under the following BRST symmetry (the bosonic symmetry after fermionization)

$$\begin{aligned} \boldsymbol{Q}_o u^I &= \psi^I, & \boldsymbol{Q}_o F_I^{(2)} &= -d\eta_I, \\ \boldsymbol{Q}_o A^I &= -d\psi^I, & \boldsymbol{Q}_o H_I &= -d\chi_I + \eta_I, \end{aligned} \qquad (3.5)$$

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where ψ^I and χ_I are 0-forms taking values in $u^*(\Pi TX)$ and $u^*(\Pi T^*X)$, respectively, with $U = 1$ while η_I are 1-forms taking value in $u^*(\Pi T^*X)$ with $U = 1$. We have $\boldsymbol{Q}_o I_o = \int_{\partial N}(\eta_I \wedge du^I + \psi^I dH_I)$. The boundary conditions are such that $\psi^I(x) = 0$ and $\eta_I(x)$ vanish along the directions tangent to ∂N, while $H_I(x) = 0$ along the direction normal to ∂N for $x \in \partial N$. The above BRST transformation laws should be completed by demanding $\boldsymbol{Q}_o^2 = 0$;

$$\boldsymbol{Q}_o \psi^I = 0, \qquad \boldsymbol{Q}_o \eta_I = dF_I, \qquad \boldsymbol{Q}_o \chi_I = F_I, \qquad \boldsymbol{Q}_o F_I = 0. \tag{3.6}$$

Note that we introduce a new scalar bosonic field F_I taking values in $u^*(T^*X)$ with $U = 2$.[12] The boundary condition is that $F_I(x) = 0$ for $x \in \partial \Sigma$.

Now all those fields appearing above are regarded as "fields". One then introduces "anti-fields" as follows,

$$
\begin{array}{ccccccccc}
\text{Fields} & u^I & F_I & \eta_I & F_I^{(2)} & \chi_I & H_I & \psi^I & A^I \\
\text{Anti-Fields} & \eta_I^{(3)} & \rho_{(3)}^I & u_{(2)}^I & \rho^I & A_{(3)}^I & \psi_{(2)}^I & H_I^{(3)} & \chi_I^{(2)}.
\end{array} \tag{3.7}
$$

Here we used the convention that Latin letters denote bosonic (or even) fields while Greek letters denote fermionic (or odd) fields. If a "field" is a n-form on N its "anti-field" is a $(3 - n)$-form. The ghost numbers U of a "field" ϕ and its "anti-field" $\overline{\phi}$ are relates as $U(\overline{\phi}) = -1 - U(\phi)$. Now we follows the usual steps to find a BV master action functional. The resulting theory is described in the following subsection.

3.1   BV quantized membrane without background

We start from the total space of ΠT^*X, with local coordinates $(\{u^I\}, \{\chi_I\})$. Next, we consider the total space of $\Pi T(\Pi T^*X)$, with local coordinates $(\{\psi^I\}, \{F_I\})$ on the the fiber. We assign degrees or ghost numbers $U = (0, 1, 1, 2)$ to $(u^I, \chi_I, \psi^I, F_I)$. Now we define the following *even* symplectic structure on the space $\Pi T(\Pi T^*X)$ with degree $U = 2$:

$$\omega = dF_I du^I + d\psi^I d\chi_I. \tag{3.8}$$

We note that the total space of $\Pi T(\Pi T^*X)$ can be identified with the total space of $T^*[2](\Pi T^*X)$. Then the degree $U = 2$ symplectic structure ω

---

[12] From $\boldsymbol{Q}_o^2 F_I^{(2)} = -d\left(\boldsymbol{Q}_o \eta_I\right) = 0$, we see that the general solution for $\boldsymbol{Q}_o \eta_I$ is an exact 1-form. The moral is that wherever there is an ambiguity, which is actually a gauge symmetry, there should be a new field (or ghost for ghost).

can be identified with the canonical symplectic form on $T^*[2](\Pi T^*X)$ as a "cotangent bundle".

Based on the even symplectic structure ω we define an even Poisson bracket among functions in $\big(\{\phi^I\}, \{\chi_I\}|\{\psi^I\}, \{F_I\}\big)$, i.e. functions on $\Pi T(\Pi T^*X)$, by the formula

$$\{\gamma_1, \gamma_2\}_P = \frac{\partial \gamma_1}{\partial F_I}\frac{\partial \gamma_2}{\partial u^I} - (-1)^{|\gamma_1||\gamma_2|}\frac{\partial \gamma_2}{\partial F_I}\frac{\partial \gamma_1}{\partial u^I} + \frac{\partial \gamma_1}{\partial \psi^I}\frac{\partial \gamma_2}{\partial \chi_I} - (-1)^{|\gamma_1||\gamma_2|}\frac{\partial \gamma_2}{\partial \psi^I}\frac{\partial \gamma_1}{\partial \chi_I},$$
(3.9)

where $|\gamma|$ denotse the degree U of γ. We note that the graded Poisson bracket has degree $U = -2$, i.e., $|\{\gamma_1, \gamma_2\}_P| = |\gamma_1| + |\gamma_2| - 2$. It is not difficult to check the following properties:

$$\{\gamma_1, \gamma_2\}_P = -(-1)^{|\gamma_1||\gamma_2|}\{\gamma_2, \gamma_1\}_P,$$
$$\{\gamma_1, \gamma_2\gamma_3\}_P = \{\gamma_1, \gamma_2\}_P\gamma_3 + (-1)^{|\gamma_1||\gamma_2+1|}\gamma_2\{\gamma_1, \gamma_3\}_P, \qquad (3.10)$$
$$\{\gamma_1, \{\gamma_2, \gamma_3\}_P\}_P = \{\{\gamma_1, \gamma_2\}_P, \gamma_3\}_P + (-1)^{|\gamma_1||\gamma_2|}\{\gamma_2, \{\gamma_1, \gamma_3\}_P\}_P.$$

The second relation above (the Leibniz law) implies that the bracket behaves as a derivation of the ordinary product of functions. Such a product is (super)-commutative and associative and carries degree 0. Thus functions on $\Pi T(\Pi T^*X)$ form an algebra with a degree -2 Poisson bracket, a (super)-commutative associative product with degree 0 and a vanishing differential (no differential operator). We call such an algebra a cohomological 3-algebra.

On the space $\Pi T(\Pi T^*X)$ we have a canonical nilpotent odd vector Q_o with $U = 1$

$$Q_o = \psi^I\frac{\partial}{\partial u^I} + F_I\frac{\partial}{\partial \chi_I}, \qquad (3.11)$$

owhich riginates from the exterior derivative on ΠT^*X. We find that Q_o is the Hamiltonian vector of the following odd function

$$h = F_I\psi^I \qquad (3.12)$$

carrying degree $U = 3$. Note that an odd (even) function on $\Pi T(\Pi T^*X)$ has odd (even) Hamiltonian vector since our symplectic form is even. We have

$$\{h, h\}_P = 0, \qquad \{h, \ldots\}_P = Q_o, \qquad Q_o^2 = 0. \qquad (3.13)$$

The BV quantized topological open membrane is a theory of maps

$$\phi : \Pi T N \to \Pi T(\Pi T^*X), \qquad (3.14)$$

where $\Pi T N$ is the parity change of the total space of tangent bundle TN of N. We denote local coordinates on $\Pi T N$ by $(\{x^\mu\}, \{\theta^\mu\})$, $\mu = 1, 2, 3$. We parameterize a map by functions

$$(\boldsymbol{u}^I, \boldsymbol{\chi}_I, \boldsymbol{\psi}^I, \boldsymbol{F}_I) := (u^I, \chi_I, \psi^I, F_I)(x^\mu, \theta^\mu). \tag{3.15}$$

Now we consider the space $\mathcal{A}$ of all maps (3.14). For any function γ on $\Pi T(\Pi T^* X)$ we denote the corresponding function of $(\{x^\mu\}, \{\theta^\mu\})$ by $\boldsymbol{\gamma}$, i.e., $\boldsymbol{\gamma} = \boldsymbol{\phi}^*(\gamma) = \gamma(x^\mu, \theta^\mu)$. We have an expansion

$$\boldsymbol{\gamma} = \gamma(x^\sigma) + \gamma_\mu(x^\sigma)\theta^\mu + \frac{1}{2}\gamma_{\mu\nu}(x^\sigma)\theta^\mu\theta^\nu + \frac{1}{3!}\gamma_{\mu\nu\rho}(x^\sigma)\theta^\mu\theta^\nu\theta^\rho. \tag{3.16}$$

We denote $\gamma^{(n)} = \frac{1}{n!}\gamma_{m_1\dots\mu_n}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_n}$, where $n = 0, \dots, 3$. We obtain functions on $\Pi T\mathcal{A}$ by $\int_{c_n}\gamma^{(n)} \equiv \int_{c_n}d^n\theta\,\boldsymbol{\gamma}$, where c_n is a n-dimensional cycle in N. We see that $\int_M d^3\theta\,\boldsymbol{\gamma}$ is an even (odd) function on $\mathcal{A}$ if γ is an odd (even) function on $\Pi T(\Pi T^* X)$, with the degree shifted by -3.

On the space $\mathcal{A}$ of all fields we have the following odd symplectic structure $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \int_N d^3\theta \left(\delta\boldsymbol{F}_I\delta\boldsymbol{u}^I + \delta\boldsymbol{\psi}^I\delta\boldsymbol{\chi}_I\right). \tag{3.17}$$

We note that a n-form field is paired with a $(3-n)$-form field. For instance the 0-form part of $\boldsymbol{u}^I$, which is even, is paired with the 3-form part of $\boldsymbol{F}_I$, which is odd. We also remark that θ^μ carry degree $U = -1$ and the degree of $\boldsymbol{\omega}$ is $U = -1$. Then we can define the BV bracket $(.,.)_{BV}$ as the odd Poisson bracket with respect to $\boldsymbol{\omega}$ among functions on $\mathcal{A}$. The degree of the BV bracket is $U = 1$. We observe that the *odd* symplectic form $\boldsymbol{\omega}$ (of degree $U = -1$) on $\mathcal{A}$ is induced from the *even* symplectic structure ω (of degree $U = 2$) on $\Pi T(\Pi T^* X)$. Similarly the BV bracket (of degree $U = 1$) among functions on $\mathcal{A}$ is induced from the even Poisson bracket $\{.,.\}_{PB}$ (of degree $U = -2$) among functions on $\Pi T(\Pi T^* X)$. We also note that the relations in (3.10) after shifting the degree by -3 become the usual relations for the BV bracket. It follows that

$$\{\gamma, \gamma\}_P = 0 \text{ if and only if } \left(\int_N d^3\theta\,\boldsymbol{\gamma}, \int_N d^3\theta\,\boldsymbol{\gamma}\right)_{BV} = 0.$$

We note that the superfields in (3.15) contain all the "fields" and "antifields" (3.7) in the BV quantization of membrane. For the explicit identifi-

cations we expand the superfields as follows:

$$\boldsymbol{u}^I = u^I + \rho^I_\mu \theta^\mu + \frac{1}{2} u^I_{\mu\nu} \theta^\mu \theta^\nu + \frac{1}{3!} \rho^I_{\mu\nu\rho} \theta^\mu \theta^\nu \theta^\sigma,$$

$$\boldsymbol{F}_I = F_I + \eta_{I\mu} \theta^\mu + \frac{1}{2} F_{I\mu\nu} \theta^\mu \theta^\nu - \frac{1}{3!} \eta_{I\mu\nu\sigma} \theta^\mu \theta^\nu \theta^\sigma,$$

$$\boldsymbol{\psi}^I = \psi^I + A^I_\mu \theta^\mu - \frac{1}{2} \psi^I_{\mu\nu} \theta^\mu \theta^\nu - \frac{1}{3!} A^I_{\mu\nu\rho} \theta^\mu \theta^\nu \theta^\sigma,$$

$$\boldsymbol{\chi}_I = \chi_I + H_{I\mu} \theta^\mu + \frac{1}{2!} \chi_{I\mu\nu} \theta^\mu \theta^\nu + \frac{1}{3!} H_{I\mu\nu\sigma} \theta^\mu \theta^\nu \theta^\sigma.$$

$$(3.18)$$

The ghost number of the superfields $(\boldsymbol{u}^I, \boldsymbol{\chi}_I, \boldsymbol{\psi}^I, \boldsymbol{F}_I)$ are $U = (0, 1, 1, 2)$, the same as the ghost numbers of $(u^I, \chi_I, \psi^I, F_I)$. Note that the ghost number of θ^μ is $U = 1$. As a differential form we write, for example, $\rho^I = \rho^I_\mu dx^\mu$, $u^I_{(2)} = \frac{1}{2} u^I_{\mu\nu} dx^\mu \wedge dx^\nu$ and $\rho^I_{(3)} = \frac{1}{3!} \rho^I_{\mu\nu\sigma} dx^\mu dx^\nu dx^\sigma$, etc. Thus the assignments of "fields" and "anti-fields" in (3.7) and the ghost numbers are consistent with our definition of the odd symplectic structure $\boldsymbol{\omega}$ (3.17) and the decompositions of the superfields. One can also check that for any function γ on $\Pi T(\Pi T^* X)$, we have

$$\triangle \int_N d^3\theta \, \phi^*(\gamma) = (-1 + 3 - 3 + 1)\mathcal{C} \int_N dv \left(\frac{\delta^2}{\delta F_I \delta u^I} + \frac{\delta^2}{\delta \psi^I \delta \chi_I} \right) \gamma^{(0)}$$

$$= 0,$$

$$(3.19)$$

where $\mathcal{C}$ is an infinite constant and dv is the volume form on N.

The BV quantized version $\boldsymbol{S}_o$ of the action functional I_o in (3.3) is given by

$$\boldsymbol{S}_o = \int_M d^3\theta \left(\boldsymbol{\psi}^I D \boldsymbol{\chi}_I + \boldsymbol{F}_I D \boldsymbol{u}^I + \boldsymbol{F}_I \boldsymbol{\psi}^I \right). \qquad (3.20)$$

The BV BRST charge $\boldsymbol{Q}_o$ corresponds to the odd Hamiltonian vector $\boldsymbol{Q}_o = (\boldsymbol{S}_o, \ldots)_{BV}$ of $\boldsymbol{S}_o$. We have

$$\boldsymbol{Q}_o = D + \phi^*(Q_o), \qquad (3.21)$$

where $D = \theta^\mu \partial_\mu$. Explicitly,

$$\boldsymbol{Q}_0 = \left(D\boldsymbol{u}^I + \boldsymbol{\psi}^I \right) \frac{\partial}{\partial \boldsymbol{u}^I} + D\boldsymbol{\psi}^I \frac{\partial}{\partial \boldsymbol{\psi}^I} + \left(D\boldsymbol{\chi}_I + \boldsymbol{F}_I \right) \frac{\partial}{\partial \boldsymbol{\chi}_I} + D\boldsymbol{F}_I \frac{\partial}{\partial \boldsymbol{F}_I}. \quad (3.22)$$

In components we see that the above BRST charge leads to the transformation laws (3.5) and (3.6) for the "fields' as well as for the "anti-fields".

It is trivial to check that S_o satisfies the quantum master equation,

$$(S_0, S_0)_{BV} = 0, \qquad \triangle S_0 = 0. \tag{3.23}$$

Now we consider the case that the boundary of N is non-empty. Then we should impose suitable boundary conditions such that $DS_o = 0$.

$$\int_{\partial N} d^2\theta \; F_I \psi^I = 0,$$

$$\int_{\partial N} d^2\theta \; \left(\psi^I D\chi_I + F_I Du^I\right) = 0. \tag{3.24}$$

The above equations are satisfied by the boundary conditions we introduced earlier. We note that the F_I enter into the action functional linearly. Thus the integration over F_I leads to a delta function like constraint

$$Du^I + \psi^I = 0. \tag{3.25}$$

Then the on-shell action functional reduces to the BV quantized boundary closed string theory without background,

$$S_o|_{\text{on shell}} = \int_{\partial N} d^2\theta \left(\chi_I Du^I\right). \tag{3.26}$$

3.2   Bulk deformations and boundary observables

Now we consider a deformation the BV action functional S_o preserving the ghost number symmetry. A deformed action functional is of the following form

$$S_\gamma = S_o + \int_N d^3\theta \; \phi^*(\gamma), \tag{3.27}$$

where γ is a function on $\Pi T(\Pi T^* X)$. The idea and procedure behind the above deformation are exactly the same as those of the string case discussed extensively in Sect. 2.2.2. The Brezin integral $\int d^3\theta$ will decrease the ghost number by $U = 3$. Thus γ should have degree $U = 3$ to preserve the ghost number symmetry. Here we will impose such a condition.

The above deformation is well-defined or admissible if S_γ satisfies the quantum master equation,

$$-\hbar\triangle S_\gamma + \frac{1}{2}\left(S_\gamma, S_\gamma\right)_{BV} = 0. \tag{3.28}$$

Since $\triangle S_\gamma = \triangle \int_N d^3\theta \; \gamma = 0$, the quantum master equation (3.28) reduces to

$$\left(S_\gamma, S_\gamma\right)_{BV} = 0. \tag{3.29}$$

The above is equivalent to the following two conditions:

$$\int_N d^3\theta \; D\boldsymbol{\phi}^*(\gamma) = 0,$$
$$\int_N d^3\theta \; \boldsymbol{\phi}^*(Q_o\gamma + \frac{1}{2}\{\gamma, \gamma\}_P) = 0. \tag{3.30}$$

Note that the boundary conditions are such that $\boldsymbol{\psi}^I(x), \boldsymbol{F}_I(x) = 0$ in directions tangent to ∂N_p for $x \in \partial N_p$. Thus a consistent deformation of the theory is determined by a degree $U = 3$ function γ on $\Pi T(\Pi T^* X)$ satisfying

$$\gamma|_{\Pi T^* X} = 0,$$
$$Q_o\gamma + \frac{1}{2}\{\gamma, \gamma\}_P = 0. \tag{3.31}$$

The first condition means that the restriction of $\gamma(u^I, \chi_I, F^I, \psi_I)$ to the base space $\Pi T^* X$ of $\Pi T(\Pi T^* X)$ vanishes. We denote the set of equivalence classes of all solution of (3.31) by $\mathfrak{M}^{cl}$. Thus the space $\mathfrak{M}^{cl}$ is isomorphic to the set of equivalence classes of all consistent bulk deformations (or backgrounds) of the membrane theory $\boldsymbol{S}_o$ preserving the ghost number symmetry U.

From now on we assume that γ is a solution of (3.31). We denote by Q_γ the degree $U = 1$ odd nilpotent Hamiltonian vector of $h + \gamma$;

$$\{h + \gamma, \ldots\}_P = Q_\gamma. \tag{3.32}$$

Explicitly

$$Q_\gamma = \left(\psi^I + \frac{\partial\gamma}{\partial F_I}\right)\frac{\partial}{\partial u^I} + \left(F_I + \frac{\partial\gamma}{\partial\psi^I}\right)\frac{\partial}{\partial\chi_I}$$
$$+ \frac{\partial\gamma}{\partial u^I}\frac{\partial}{\partial F_I} + \frac{\partial\gamma}{\partial\chi_I}\frac{\partial}{\partial\psi^I}. \tag{3.33}$$

Then the action functional $\boldsymbol{S}_\gamma$ in (3.27) has a BRST symmetry generated by the odd nilpotent vector $\boldsymbol{Q}_\gamma$ with $U = -1$ defined by $(\boldsymbol{S}_\gamma, \ldots)_{BV} = \boldsymbol{Q}_\gamma$. Equivalently[13]

$$\boldsymbol{Q}_\gamma = D + \boldsymbol{\phi}^*(Q_\gamma). \tag{3.34}$$

---

[13]We should emphasize that the BV master equation is equivalent to the condition that $\boldsymbol{Q}_\gamma^2 = 0$, which implies that $Q_\gamma^2 = 0$ and the first equation of (3.30). Thus the moduli space of the theory is defined *not* by the non-linear cohomology of Q_γ but by the non-linear cohomology of the BV BRST charge $\boldsymbol{Q}_\gamma$. The difference is precisely encoded in the

Denoting $\boldsymbol{\gamma} = \boldsymbol{\phi}^*(g(u^I, \chi_I, F_I, \psi^I)) = \gamma(\boldsymbol{u}^I, \boldsymbol{\chi}_I, \boldsymbol{F}_I, \boldsymbol{\psi}^I)$ we have

$$
\begin{aligned}
\boldsymbol{Q}_\gamma &= \left(D\boldsymbol{u}^I + \boldsymbol{\psi}^I + \frac{\partial\gamma}{\partial \boldsymbol{F}_I} \right) \frac{\partial}{\partial \boldsymbol{u}^I} + \left(D\boldsymbol{\chi}_I + \boldsymbol{F}_I + \frac{\partial\gamma}{\partial \boldsymbol{\psi}^I} \right) \frac{\partial}{\partial \boldsymbol{\chi}_I} \\
&+ \left(D\boldsymbol{F}_I + \frac{\partial\gamma}{\partial \boldsymbol{u}^I} \right) \frac{\partial}{\partial \boldsymbol{F}_I} + \left(D\boldsymbol{\psi}^I + \frac{\partial\gamma}{\partial \boldsymbol{\chi}_I} \right) \frac{\partial}{\partial \boldsymbol{\psi}^I}.
\end{aligned}
\tag{3.35}
$$

The action functional $\boldsymbol{S}_\gamma$ also has a fermionic symmetry generated by $\mathcal{K}_\mu = -\frac{\partial}{\partial\theta^\mu}$, acting on superfields, with $U = -1$. Together with $\boldsymbol{Q}_\gamma$ they satisfy the following anti-commutation relations,

$$
\{\boldsymbol{Q}_\gamma, \boldsymbol{Q}_\gamma\} = 0, \qquad \{\boldsymbol{Q}_\gamma, \mathcal{K}_\mu\} = -\partial_\mu, \qquad \{\mathcal{K}_\mu, \mathcal{K}_\nu\} = 0.
\tag{3.36}
$$

Given a $\boldsymbol{Q}_\gamma$ invariant function $A^{(0)}$ on $\mathcal{A}$, which is a 0-form on N, the n-form $A^{(n)} := \frac{1}{n!} \left(\mathcal{K}_{\mu_1} \ldots \mathcal{K}_{\mu_n} A^{(0)} \right) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}$ for $n = 1, 2, 3$ give the canonical set of solutions of the descent equations

$$
\begin{aligned}
\boldsymbol{Q}_\gamma A^{(0)} &= 0, \\
\boldsymbol{Q}_\gamma A^{(1)} + dA^{(0)} &= 0, \\
\boldsymbol{Q}_\gamma A^{(2)} + dA^{(1)} &= 0, \\
\boldsymbol{Q}_\gamma A^{(3)} + dA^{(2)} &= 0, \\
dA^{(3)} &= 0.
\end{aligned}
\tag{3.37}
$$

The above is a direct consequence of (3.34).

A BV observable in general is a function $\mathcal{O}$ on $\mathcal{A}$ satisfying $(-\hbar\triangle + \boldsymbol{Q}_\gamma)\mathcal{O} = 0$. A zero-dimensional observable $\mathcal{O}^{(0)}(x)$ can be inserted at a point in the interior or at the boundary of N. Since $\mathcal{O}^{(0)}$ is a scalar on N it is a function of the scalar components of the superfields only. It follows that $\triangle\mathcal{O}^{(0)} = 0$ and $\boldsymbol{Q}_\gamma\mathcal{O}^{(0)} = 0$. The latter, together with (3.34), implies that $\mathcal{O}^{(0)}(x)$, $x \notin \partial\Sigma$, must derive from a Q_γ cohomology class among functions on $\Pi T\Pi(T^*X)$. Recall that the boundary condition for $\psi^I(x^\mu)$ and $F_I(x^\mu)$ is that they vanish identically at the boundary. Thus a zero observable inserted at a boundary point originates from a function $f(u^I, \chi_I)$ of the base $\Pi(T^*X)$ of $\Pi T(\Pi(T^*X))$. Note that no such function can be used to

boundary degrees due to the relation

$$
\boldsymbol{Q}_\gamma \int_N d^3\theta\phi^*(\alpha) - \int_N d^3\theta\phi^*(Q_\gamma\alpha) = \int_{\partial N} d^2\theta\phi^*(\alpha)|_{\partial N}.
$$

deform the bulk action functional due to the first condition in (3.31). We may take for N a three-dimensional disk with boundary $\partial N = S^2$, having a number of marked points x_i where zero-dimensional observables are inserted. We can also consider a small circle $C_i \subset \partial \Sigma$ surrounding a marked point x_i. Then $\oint_{C_i} d^\theta f(\boldsymbol{u}^I, \boldsymbol{\chi}_I)$ defines an observable, whose expectation value does not depend on the contour. We have other observables $\int_{\partial N} d^2\theta\ f(\boldsymbol{u}^I, \boldsymbol{\chi}_I)$, which may be viewed as boundary interactions.

Now our goal is to determine the general γ with $U = 3$ satisfying (3.31). The explicit form of the above equation is easy to write down, though complicated, by the general form of a degree $U = 3$ function γ on on $\Pi T(\Pi T^* X)$.

3.2.1 Turning on a C-field

Now we consider a bulk deformation leading to the BV quantized version of the action functional of the membrane with a flat 3-form C-field (3.2). For a 3-form $C = \frac{1}{3!}c_{IJK}(u^L)du^I \wedge du^J \wedge du^K$ on X, we obtain a degree $U = 3$ function c on $\Pi T(\Pi T^*(X))$

$$c = \frac{1}{3!}c_{IJK}(u^L)\psi^I\psi^J\psi^K, \tag{3.38}$$

satisfying $c|_{\Pi T^* X} = 0$. The condition $Q_o c + \frac{1}{2}\{c, c\}_P = 0$ is equivalent to $dC = 0$. Thus we obtain the following action functional satisfying the BV master equation

$$\boldsymbol{S}_c = \int_N d^3\theta \left(D\boldsymbol{\chi}_I\boldsymbol{\psi}^I + \boldsymbol{F}_I D\boldsymbol{u}^I + \boldsymbol{F}_I\boldsymbol{\psi}^I\right) + \frac{1}{3!}\int_N d^3\theta \left(\boldsymbol{c}_{IJK}\boldsymbol{\psi}^I\boldsymbol{\psi}^J\boldsymbol{\psi}^K\right). \tag{3.39}$$

This is the BV quantized action functional of (3.2). The action functional $\boldsymbol{S}_c$ is the Hamiltonian function, $(\boldsymbol{S}_c, \dots)_{BV} = \boldsymbol{Q}_c$, of the odd vector $\boldsymbol{Q}$ given by (in terms of BRST transformation laws)

$$\begin{aligned}
\boldsymbol{Q}_c\boldsymbol{u}^I &= D\boldsymbol{u}^I + \boldsymbol{\psi}^I, \\
\boldsymbol{Q}_c\boldsymbol{\chi}_I &= D\boldsymbol{\chi}_I + \boldsymbol{F}_I + \frac{1}{2}\boldsymbol{c}_{IJK}\boldsymbol{\psi}^J\boldsymbol{\psi}^K, \\
\boldsymbol{Q}_c\boldsymbol{\psi}^I &= D\boldsymbol{\psi}^I \\
\boldsymbol{Q}_c\boldsymbol{F}_I &= D\boldsymbol{F}_I - \frac{1}{3!}\partial_I\boldsymbol{c}_{JKL}\boldsymbol{\psi}^J\boldsymbol{\psi}^K\boldsymbol{\psi}^L.
\end{aligned} \tag{3.40}$$

On-shell we can eliminate $\boldsymbol{F}_I$ and $\boldsymbol{\psi}^I$ using the $\boldsymbol{F}_I$ equation of motion;

$$\boldsymbol{S}_c|_{\text{on shell}} = -\frac{1}{3!}\int_N d^3\theta \left(\boldsymbol{c}_{IJK}D\boldsymbol{u}^I D\boldsymbol{u}^J D\boldsymbol{u}^K\right) + \int_{\partial N} d^2\theta\ \boldsymbol{\chi}_I D\boldsymbol{u}^I. \tag{3.41}$$

Now consider an arbitrary function $f(u^I, \chi_I)$ on the base $\Pi T^* X$ of $\Pi T(\Pi T^* X)$. The corresponding function $\boldsymbol{f}$ of the superfields has an expansion

$$\boldsymbol{f} = f^{(0)}(x^\mu) + \dots. \tag{3.42}$$

Inserting $f^{(0)}(x_i)$ at a boundary puncture x_i we find $\boldsymbol{Q}_c f^{(0)} = 0$. It also follows that $\oint_{C_i} d\theta \boldsymbol{f}$ and $\int_{\partial N} d\theta \boldsymbol{f}$ are BV observables.

Now we discuss more general deformations. We consider the following non-linear transformations of the fiber coordinates of $\Pi T(\Pi T^* X) \to \Pi T^* X$,

$$\begin{aligned} F_I \to F'_I &:= F_I - \frac{\partial \pi}{\partial u^I}, \\ \psi^I \to \psi'^I &:= \psi^I - \frac{\partial \pi}{\partial \chi_I}, \end{aligned} \tag{3.43}$$

with u^I and χ unchanged. Here π is an arbitrary degree $U = 2$ function on the base space $\Pi T^* X$ of $\Pi T(\Pi T^* X)$, i.e., $\pi = \frac{1}{2} b^{IJ}(u^L)\chi_I\chi_J$. Under the above transformation we have

$$h + c \to h + \gamma,$$

where

$$\begin{aligned} \gamma = &- F_I \frac{\partial \pi}{\partial \chi_I} - \frac{\partial \pi}{\partial u^I}\psi^I + \frac{1}{2}[\pi, \pi]_S \\ &+ \frac{1}{3!}c_{IJK}\left(\psi^I - \frac{\partial \pi}{\partial \chi_I}\right)\left(\psi^J - \frac{\partial \pi}{\partial \chi_J}\right)\left(\psi^K - \frac{\partial \pi}{\partial \chi_K}\right) \end{aligned} \tag{3.44}$$

It can be easily shown that $\{h + \gamma, h + \gamma\}_P = 0$ if and only if $dC = 0$. The above class of solutions of (3.31) is determined by an element of $H^3(X)$ and a bi-vector on X. Now the condition $\gamma|_{\Pi T^* X} = 0$ implies that

$$\frac{1}{2}[\pi, \pi]_S - \frac{1}{3!}c_{IJK}\frac{\partial \pi}{\partial \chi_I}\frac{\partial \pi}{\partial \chi_I}\frac{\partial \pi}{\partial \chi_I} = 0. \tag{3.45}$$

More explicitly

$$h^{LMN} - c_{IJK}b^{IL}b^{JM}b^{KN} = 0, \tag{3.46}$$

where $h^{IJK} = b^{LI}\partial_L b^{JK} + cyclic$. Such a pair (c, π) leads to the following degree $U = 3$ function on $T^*\Pi(T^* X)$ satisfying (3.31),

$$\begin{aligned} \gamma = &- b^{IJ}F_I\chi_J - \frac{1}{2}\left(\partial_I b^{JK} + c_{IMN}b^{MJ}b^{NK}\right)\chi_J\chi_K\psi^I \\ &- \frac{1}{2}c_{IJK}b^{IL}\chi_L\psi^J\psi^K + \frac{1}{3!}c_{IJK}\psi^I\psi^J\psi^K. \end{aligned} \tag{3.47}$$

It follows that the action functional

$$
\boldsymbol{S}_\gamma = \int_N d^3\theta \Big(D\boldsymbol{\chi}_I\boldsymbol{\psi}^I + \boldsymbol{F}_I D\boldsymbol{u}^I + \boldsymbol{F}_I\boldsymbol{\psi}^I + \frac{1}{3!}\boldsymbol{c}_{IJK}\boldsymbol{\psi}^I\boldsymbol{\psi}^J\boldsymbol{\psi}^K - \boldsymbol{b}^{IJ}\boldsymbol{F}_I\boldsymbol{\chi}_J
$$
$$
- \frac{1}{2}\boldsymbol{c}_{IJK}\boldsymbol{b}^{IL}\boldsymbol{\chi}_L\boldsymbol{\psi}^J\boldsymbol{\psi}^K - \frac{1}{2}\left(\partial_I\boldsymbol{b}^{JK} + \boldsymbol{c}_{IMN}\boldsymbol{b}^{MJ}\boldsymbol{b}^{NK}\right)\boldsymbol{\chi}_J\boldsymbol{\chi}_K\boldsymbol{\psi}^I\Big)
$$
$$
(3.48)
$$

satisfies the quantum BV master equation provided that $dC = 0$.

The BV BRST transformation laws, with the condition (3.46), are given by

$$
\boldsymbol{Q}\boldsymbol{u}^I = D\boldsymbol{u}^I + \boldsymbol{\psi}^I - \boldsymbol{b}^{IJ}\boldsymbol{\chi}_J,
$$
$$
\boldsymbol{Q}\boldsymbol{\chi}_I = D\boldsymbol{\chi}_I + \boldsymbol{F}_I - \frac{1}{2}\partial_I\boldsymbol{b}^{JK}\boldsymbol{\chi}_J\boldsymbol{\chi}_K + \frac{1}{2}\boldsymbol{c}_{IJK}\left(\boldsymbol{\psi}^J - \boldsymbol{b}^{JM}\boldsymbol{\chi}_M\right)\left(\boldsymbol{\psi}^K - \boldsymbol{b}^{KN}\boldsymbol{\chi}_N\right),
$$
$$
\boldsymbol{Q}\boldsymbol{\psi}^I = D\boldsymbol{\psi}^I - \boldsymbol{b}^{IJ}\boldsymbol{F}_J - \partial_K\boldsymbol{b}^{IJ}\boldsymbol{\psi}^K\boldsymbol{\chi}_J - \frac{1}{2}\boldsymbol{h}^{IJK}\boldsymbol{\chi}_J\boldsymbol{\chi}_K
$$
$$
- \frac{1}{2}\boldsymbol{c}_{LJK}\boldsymbol{b}^{LI}\left(\boldsymbol{\psi}^J - \boldsymbol{b}^{JM}\boldsymbol{\chi}_M\right)\left(\boldsymbol{\psi}^K - \boldsymbol{b}^{KN}\boldsymbol{\chi}_N\right),
$$
$$
\boldsymbol{Q}\boldsymbol{F}_I = D\boldsymbol{F}_I + \partial_I\boldsymbol{b}^{JK}\boldsymbol{F}_J\boldsymbol{\chi}_K + \frac{1}{2}\partial_I\partial_K\boldsymbol{b}^{LJ}\boldsymbol{\psi}^K\boldsymbol{\chi}_L\boldsymbol{\chi}_J + \frac{1}{3!}\partial_I\boldsymbol{h}^{JKL}\boldsymbol{\chi}_J\boldsymbol{\chi}_K\boldsymbol{\chi}_L
$$
$$
- \frac{1}{3!}\partial_I\boldsymbol{c}_{PJK}\left(\boldsymbol{\psi}^P - \boldsymbol{b}^{PL}\boldsymbol{\chi}_L\right)\left(\boldsymbol{\psi}^J - \boldsymbol{b}^{JM}\boldsymbol{\chi}_M\right)\left(\boldsymbol{\psi}^K - \boldsymbol{b}^{KN}\boldsymbol{\chi}_N\right)
$$
$$
+ \frac{1}{2}\boldsymbol{c}_{PJK}\partial_I\boldsymbol{b}^{PL}\boldsymbol{\chi}_L\left(\boldsymbol{\psi}^J - \boldsymbol{b}^{JM}\boldsymbol{\chi}_M\right)\left(\boldsymbol{\psi}^K - \boldsymbol{b}^{KN}\boldsymbol{\chi}_N\right).
$$
$$
(3.49)
$$

We have $\boldsymbol{Q}_\gamma^2 = 0$, which follows from $(\boldsymbol{S}_\gamma, \boldsymbol{S}_\gamma)_{BV} = 0$.

Now we examine the use of the above deformation. From the $\boldsymbol{F}_I$ equation of motion

$$
D\boldsymbol{u}^I - \boldsymbol{b}^{IJ}\boldsymbol{\chi}_J + \boldsymbol{\psi}^I = 0 \tag{3.50}
$$

we can eliminate $\boldsymbol{\psi}^I$ from the action functional, leading to the following on-shell action functional

$$
\boldsymbol{S}_\gamma|_{\text{on shell}} = -\frac{1}{3!}\int_N d^3\theta\left(\boldsymbol{c}_{IJK}D\boldsymbol{u}^I D\boldsymbol{u}^J D\boldsymbol{u}^K\right)
$$
$$
+ \int_{\partial N} d^2\theta\left(\boldsymbol{\chi}_I D\boldsymbol{u}^I + \frac{1}{2}\boldsymbol{b}^{IJ}\boldsymbol{\chi}_I\boldsymbol{\chi}_J\right).
$$
$$
(3.51)
$$

We obtain a boundary string theory in an arbitrary bivector background, while a closed 3-form C-field is coupled to the membrane in the bulk. Note that the boundary action functional may be viewed as the closed string version of the Catteneo-Felder model. Recall that such an action functional

satisfies the quantum master equation (but with a different BV bracket) if and only if π is Poisson. We may identify the open membrane theory defined by the action functional $\boldsymbol{S}_\gamma$ (3.48) together with the condition (3.46) as the off-shell closed string theory coupled to an arbitrary B-field.

Now we assume that the bi-vector is non-degenerate, and therefore has an inverse. We then have a corresponding 2-form or anti-symmetric tensor field on X, $B = \frac{1}{2}b_{IJ}du^I \wedge du^J$. Then the condition (3.46) implies that $C = dB$. Hence the 3-form C is the field strength of the B-field.

3.3 The first approximation

In this subsection we discuss the first approximation of the path integral for a manifold X with $c_1(X) = 0$. Our presentation will be indirect and the actual path integral calculations will appear elsewhere [37].

The first order problem can be viewed as a "quantization" of $\Pi T(\Pi T^*X)$ viewed as a classical phase space with respect to the even symplectic structure ω in (3.8). Here we regard (u^I, χ_I) as the canonical coordinates and (F_I, ψ^I) as the conjugate momenta. Now recall that the bulk term is determined by the function $h + \gamma$ on $\Pi T\Pi(T^*X)$

$$
\begin{aligned}
h + \gamma = &- b^{IJ}F_I\chi_J - \frac{1}{2}\left(\partial_I b^{JK} + c_{IMN}b^{MJ}b^{NK}\right)\chi_J\chi_K\psi^I \\
&+ F_I\psi^I - \frac{1}{2}c_{IJK}b^{IL}\chi_L\psi^J\psi^K + \frac{1}{3!}c_{IJK}\psi^I\psi^J\psi^K
\end{aligned}
\tag{3.52}
$$

satisfying the condition (3.45). By "quantization" of $\Pi T(\Pi T^*X)$ we mean the following replacements

$$
F_I \to -\hbar\frac{\partial}{\partial u^I}, \qquad \psi^I \to -\hbar\frac{\partial}{\partial \chi_I},
\tag{3.53}
$$

where $\hbar$ is regarded as a formal parameter with $U = 2$. From $(h + \gamma)/\hbar$ we obtain the following differential operators acting on functions on ΠT^*X;

$$
\mathcal{D} = \mathcal{D}_1 + \hbar\mathcal{D}_2 + \hbar^2\mathcal{D}_3,
\tag{3.54}
$$

where

$$
\begin{aligned}
\mathcal{D}_1 &= b^{IJ}\chi_J\frac{\partial}{\partial u^I} + \frac{1}{2}\left(\partial_I b^{JK} + c_{IMN}b^{MJ}b^{NK}\right)\chi_J\chi_K\frac{\partial}{\partial \chi_I}, \\
\mathcal{D}_2 &= \frac{\partial^2}{\partial u^I\partial\chi_I} - \frac{1}{2}c_{IJK}b^{IL}\chi_L\frac{\partial^2}{\partial\chi_J\partial\chi_K}, \\
\mathcal{D}_3 &= \frac{1}{3!}c_{IJK}\frac{\partial^3}{\partial\chi_I\partial\chi_J\partial\chi_K}.
\end{aligned}
\tag{3.55}
$$

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Note that $\mathcal{D}_i$ is an order i differential operator and has degree $U = 3 - 2i$. Now the conditions to satisfy the BV master equation, $dC = 0$ and (3.45), imply that $\mathcal{D}^2 = 0$;

$$\mathcal{D}_1^2 = 0,$$
$$\mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2\mathcal{D}_1 = 0,$$
$$\mathcal{D}_1\mathcal{D}_3 + \mathcal{D}_3\mathcal{D}_1 + \mathcal{D}_2^2 = 0, \qquad (3.56)$$
$$\mathcal{D}_2\mathcal{D}_3 + \mathcal{D}_3\mathcal{D}_2 = 0,$$
$$\mathcal{D}_3^2 = 0.$$

The differential operators above defines various structures on the algebra $\mathcal{O}(\Pi T^* X)$ of functions on $\Pi T^* X$ (multivectors on X). For $c = \pi = 0$, therefore $\gamma = 0$, we have $\mathcal{D}_1 = \mathcal{D}_3 = 0$, and $\mathcal{D}_2 = \triangle$ generates the Schouten-Nijenhuis bracket $[,.,]_S$ on functions on $\Pi T^* X$. Together with the ordinary product we get the cohomological 2-algebra or GBV-algebra of X. For $c = 0$ the condition (3.45) implies that $[\pi, \pi]_s = 0$. Now we have $\mathcal{D}_1 = Q_\pi \equiv [\pi, \ldots]_S$, $\mathcal{D}_2 = \triangle$ and $\mathcal{D}_3 = 0$. Then $\mathcal{D}_1$ induces a differential on the cohomological 2-algebra. Forgetting the product we have a structure of dgLa on $\mathcal{O}(\Pi T^* X)$, or in general a structure of dGBV algebra.

Akman [1] (see also [10]), motivated by VOSA and generalizing Koszul, introduced the concept of higher order differential operators on a general superalgebra. Using such a differential operator, say D, he considered the following recursive definition of higher brackets,

$$\Phi_1(a) = D(a),$$
$$\Phi_2(a, b) = D(a \cdot b) - (Da) \cdot b - (-1)^{|a||D|} a \cdot \mathcal{D}(b),$$
$$\Phi_3(a, b, c) = \Phi_2(a, b \cdot c) - \Phi_2(a, b) \cdot c - (-1)^{|b|(|a|+|D|)} b \cdot \Phi_2(a, c),$$
$$\vdots$$
$$\Phi_{r+1}(a_1, \ldots, a_{r+1}) = \Phi_r(a_1, \ldots, a_r \cdot a_{r+1}) - \Phi_r(a_1, \ldots, a_r) \cdot a_{r+1}$$
$$- (-1)^{|a_r|(|a_1|+\ldots+|a_{r-1}|+|D|)} a_r \cdot \Phi_r(a_1, \ldots, a_{r-1}, a_{r+1}),$$
$$(3.57)$$

such that for D of order $\leq r$ Φ_{r+1} vanishes identically. He examined the general properties of those higher brakets.

Using the above we define the following 2- and 3-brackets associated with $\mathcal{D}$ among functions on $\Pi T^* X$

$$[a, b] = (-1)^{|a|} \Phi_2(a, b),$$
$$[a, b, c] = (-1)^{|b|} \Phi_3(a, b, c), \qquad (3.58)$$

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while the higher brackets all vanish. Explicitly

$$
\begin{aligned}
[a, b] =& \hbar \frac{\partial a}{\partial \chi_I} \frac{\partial b}{\partial u^I} - \frac{\hbar}{2} c_{IJK} b^{IL} \chi_L \frac{\partial a}{\partial \chi_J} \frac{\partial b}{\partial \chi_K} \\
& - (-1)^{(|a|+1)(|b|+1)} \times (a \leftrightarrow b),
\end{aligned}
\tag{3.59}
$$

$$
[a, b, c] = \hbar^2 c_{IJK} \frac{\partial a}{\partial \chi_I} \frac{\partial b}{\partial \chi_J} \frac{\partial c}{\partial \chi_K}.
$$

These brackets have the following properties:

(i) super-commutativity,

$$
[a, b] = -(-1)^{(|a|+1)(|b|+1)} [b, a]
\tag{3.60}
$$

(ii) deformed Leibniz law,

$$
[a, bc] - [a, b]c - (-)^{(|a|+1)|b|} b[a, c] = (-1)^{|a|+|b|} [a, b, c].
\tag{3.61}
$$

(ii) derivation,

$$
\mathcal{D}[a, b] = [\mathcal{D}a, b] + (-1)^{|a|+1} [a, \mathcal{D}b].
\tag{3.62}
$$

(iii) Jacobi identity up to homotopy,

$$
\begin{aligned}
[[a, b], c] + & (-1)^{(|a|+1)(|b|+1)} [b, [a, c]] - [a, [b, c]] \\
= & \mathcal{D}[a, b, c] + (-1)^{|b|} [\mathcal{D}a, b, c] \\
& + (-1)^{|a|+|b|} [a, \mathcal{D}b, c] + (-1)^{|a|} [a, b, \mathcal{D}c].
\end{aligned}
\tag{3.63}
$$

Now we return to some special cases. For $c = \pi = 0$ the 2-bracket $[.,.]$ becomes the Schouten-Nijenhuis bracket. Together with the product we have

$$
\begin{aligned}
[a, b] - (-1)^{|a|} \triangle(ab) + (-1)^{|a|} (\triangle a)b + a(\triangle b) &= 0, \\
[a, b] + (-1)^{(|a|+1)(|b|+1)} [b, a] &= 0, \\
[a, bc] - [a, b]c - (-)^{(|a|+1)|b|} b[a, c] &= 0, \\
[[a, b], c] + (-1)^{(|a|+1)(|b|+1)} [b, [a, c]] - [a, [b, c]] &= 0, \\
\triangle[a, b] - [\triangle a, b] - (-1)^{|a|+1} [a, \triangle b] &= 0.
\end{aligned}
\tag{3.64}
$$

For $c = 0$ with Poisson bi-vector π we have the Schouten-Nijenhuis bracket above and the differential Q_π satisfying;

$$
\begin{aligned}
Q_\pi^2 &= 0, \\
Q_\pi[a, b] - [Q_\pi a, b] - (-1)^{|a|+1} [a, Q_\pi b] &= 0.
\end{aligned}
\tag{3.65}
$$

Thus in the $c = 0$ case we have the usual structue of a dGVB algebra for a Poisson manifold X with $c_1(X) = 0$ on $\mathcal{O}(\Pi T^* X)$.

Hence the actual deformation comes from non-zero c. We remark that Kravchenko defined a BV_∞-algebra by a sum of differentials of degree $3 - 2i$ and order i, whose square is zero. Hence the deformed algebra $\mathcal{O}(\Pi T^* X)$ of functions on $\Pi T^* X$ induced by the differential operator $\mathcal{D}$ in (3.54) has a structure of BV_∞-algebra. It has a L_∞ structure with brackets and a first order differential.

3.4 Generalization

We can relax the requirement that the deformation terms should have ghost number $U = 0$, and allow them to have any even ghost number. Now we let $C_k = \frac{1}{(2k+1)!} c_{I_1 \dots I_{2k+1}} du^{I_1} \wedge \dots \wedge du^{I_{2k+1}}$ be a closed $(2k+1)$-form, $k = 0, 1, \dots$ on X;

$$dC_k = 0. \tag{3.66}$$

Then we have the corresponding degree $U = 2k+1$ function c_k on $\Pi(T\Pi T^* X)$;

$$c_k = \frac{1}{(2k+1)!} c_{I_1 \dots I_{2k+1}} \psi^{I_1} \dots \psi^{I_{2k+1}}, \tag{3.67}$$

satisfying

$$\sum_k c_k|_{\Pi T^* X} = 0,$$

$$Q_o \sum_k c_k + \frac{1}{2} \left\{ \sum_k c_k, \sum_k c_k \right\}_P = 0, \tag{3.68}$$

Now we let π_ℓ be an arbitrary degree $U = 2\ell$ function on the base of $\Pi T(\Pi T^* X) \to \Pi T^* X$, i.e., $\pi_\ell = \frac{1}{(2\ell)!} b^{I_1 \dots I_{2\ell}}(u^L) \chi_{I_1} \dots \chi_{I_{2\ell}}$. Then we consider the following non-linear transformations of the fiber coordinates of $\Pi T(\Pi T^* X) \to \Pi T^* X$,

$$F_I \to F_I' := F_I - \sum_\ell \frac{\partial \pi_\ell}{\partial u^I},$$

$$\psi^I \to \psi'^I := \psi^I - \sum_\ell \frac{\partial \pi_\ell}{\partial \chi_I}, \tag{3.69}$$

with u^I and χ unchanged. Under the above transformation we have

$$h + \sum c_k \to h + \Gamma, \tag{3.70}$$

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where

$$\Gamma = -\sum_k \left(F_I \frac{\partial \pi_k}{\partial \chi_I} + \frac{\partial \pi_k}{\partial u^I} \psi^I \right) + \frac{1}{2} \left[\sum_k \pi_k, \sum_k \pi_k \right]_S$$
$$+ \sum_k \frac{1}{(2k+1)!} c_{I_1 \dots I_{2k+1}} \left(\psi^{I_1} - \sum_\ell \frac{\partial \pi_\ell}{\partial \chi_{I_1}} \right) \dots \left(\psi^{I_{2k+1}} - \sum_\ell \frac{\partial \pi_\ell}{\partial \chi_{I_{2k+1}}} \right) \tag{3.71}$$

It can be easily shown that $Q_o \Gamma + \frac{1}{2} \{\Gamma, \Gamma\}_P = 0$ if and only if $dC_k = 0$ for all k. Now the condition $\Gamma|_{\Pi T^* X} = 0$ implies that

$$\frac{1}{2} \left[\sum_\ell \pi_\ell, \sum_\ell \pi_\ell \right]_S - \sum_k \frac{1}{(2k+1)!} c_{I_1 \dots I_{2k+1}} \left(\frac{\sum_\ell \partial \pi_\ell}{\partial \chi_{I_1}} \right) \dots \left(\frac{\sum_\ell \partial \pi_\ell}{\partial \chi_{I_{2k+1}}} \right) = 0. \tag{3.72}$$

Assuming the above conditions we get the following action functional

$$\boldsymbol{S}_\Gamma = \boldsymbol{S}_o + \int_N d^3\theta \ \boldsymbol{\phi}^*(\Gamma), \tag{3.73}$$

satisfying the BV master equation. Using the $\boldsymbol{F}_I$ equation of motion

$$D\boldsymbol{u}^I - \sum_\ell \frac{\partial \pi_\ell}{\partial \chi_I} + \boldsymbol{\psi}^I = 0, \tag{3.74}$$

we have the following on-shell action functional

$$\boldsymbol{S}_\Gamma|_{\text{on shell}} = -\sum_k \frac{1}{(2k+1)!} \int_N d^3\theta \left(\boldsymbol{c}_{I_1 \dots I_{2k+1}} D\boldsymbol{u}^{I_1} \dots D\boldsymbol{u}^{I_{2k+1}} \right)$$
$$+ \int_{\partial N} d^2\theta \left(\boldsymbol{\chi}_I D\boldsymbol{u}^I + \sum_\ell \frac{1}{(2\ell)!} \boldsymbol{b}^{I_1 \dots I_{2\ell}} \boldsymbol{\chi}_{I_1} \dots \boldsymbol{\chi}_{I_{2\ell}} \right). \tag{3.75}$$

We may also allow for arbitrary ghost numbers and the most general form for the solutions. Such a case may be used to determine explicit quasi-isomorphism of the 3-algebra.

3.5 Other boundary conditions

In this subsection we consider some variants of the topological open membrane by changing the boundary conditions. There can be more general boundary conditions than mentioned here. We pick two of them, which are relevant to mirror symmetry.

3.5.1 A boundary conditions

Now we consider the following general action functional

$$\boldsymbol{S}_\Gamma = \int_N d^3\theta \Big(\boldsymbol{\psi}^I D\boldsymbol{\chi}_I + \boldsymbol{F}_I D\boldsymbol{u}^I + \boldsymbol{F}_I \boldsymbol{\psi}^I + \phi^*(\Gamma) \Big), \qquad (3.76)$$

where Γ is defined by (3.71) with the condition (3.66). Then the above action functional satisfies the BV master equation if and only if

$$\int_N d^3\theta \, D\Big(\boldsymbol{\psi}^I D\boldsymbol{\chi}_I + \boldsymbol{F}_I D\boldsymbol{u}^I + \boldsymbol{F}_I \boldsymbol{\psi}^I + \phi^*(\Gamma) \Big) = 0. \qquad (3.77)$$

Thus we may exchange the boundary conditions of $\boldsymbol{\chi}_I$ and $\boldsymbol{\psi}^I$, such that $\chi_I(x) = 0$, and $H_I(x) = 0$ along the direction tangent to $\partial\Sigma$, while $A^I(x) = 0$ along the direction normal to $\partial\Sigma$ for $x \in \partial\Sigma$. Then we must set $c_k = 0$ for all k instead of (3.72) to satisfy the master equation. Hence the following action functional satisfies the BV master equation

$$\begin{aligned}
\boldsymbol{S} = &\int_N d^3\theta \Big(\boldsymbol{\psi}^I D\boldsymbol{\chi}_I + \boldsymbol{F}_I D\boldsymbol{u}^I + \boldsymbol{F}_I \boldsymbol{\psi}^I \Big) \\
&- \sum_k \int_N d^3\theta \Big(\boldsymbol{F}_I \frac{\partial \boldsymbol{\pi}_k}{\partial \boldsymbol{\chi}_I} + \frac{\partial \boldsymbol{\pi}_k}{\partial \boldsymbol{u}^I} \boldsymbol{\psi}^I \Big) \\
&+ \frac{1}{2} \int_N d^3\theta \Big(\phi^* \Big(\Big[\sum_k \pi_k, \sum_k \pi_k \Big]_S \Big) \Big).
\end{aligned} \qquad (3.78)$$

From the point of view of the target superspace, we are replacing $\Pi T(\Pi T^* X)$ with $\Pi T^*(\Pi T X)$. In general $T^* X \to T X$ is an isomorphism only when X is a symplectic manifold. In our case $\Pi T(\Pi T^* X) \to \Pi T^*(\Pi T X)$ is an isomorphism since our even symplectic structure ω does not see the difference. Now the boundary observables of the theory are derived from functions on $\Pi T X$, which are differential forms on X. We denote by $\mathcal{O}(\Pi T X)$ the algebra of functions on $\Pi T X$, which has the ordinary product (the wedge product).

Now we consider the role of the bulk deformations to first approximation. For simplicity we only consider the deformations preserving the ghost number symmetry;

$$\begin{aligned}
\boldsymbol{S} = &\int_N d^3\theta \Big(\boldsymbol{\psi}^I D\boldsymbol{\chi}_I + \boldsymbol{F}_I D\boldsymbol{u}^I \Big) \\
&+ \int_N d^3\theta \Big(\boldsymbol{F}_I \boldsymbol{\psi}^I - \boldsymbol{b}^{IJ} \boldsymbol{F}_I \boldsymbol{\chi}_J + \frac{1}{2} \partial_I \boldsymbol{b}^{JK} \boldsymbol{\psi}^I \boldsymbol{\chi}_J \boldsymbol{\chi}_K + \frac{1}{3!} \boldsymbol{h}^{IJK} \boldsymbol{\chi}_I \boldsymbol{\chi}_J \boldsymbol{\chi}_K \Big).
\end{aligned} \qquad (3.79)$$

Then we obtain the following differential operators

$$\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_1 + \hbar\widetilde{\mathcal{D}}_2 + \hbar^2\widetilde{\mathcal{D}}_3 \qquad (3.80)$$

where

$$\begin{aligned}
\widetilde{\mathcal{D}}_1 &= \psi^I \frac{\partial}{\partial u^I}, \\
\widetilde{\mathcal{D}}_2 &= b^{IJ} \frac{\partial^2}{\partial u^I \partial \psi^J} + \frac{1}{2}\partial_I b^{JK}\psi^I \frac{\partial^2}{\partial \psi^J \partial \psi^K}, \\
\widetilde{\mathcal{D}}_3 &= \frac{1}{3!}h^{IJK}\frac{\partial^3}{\partial \psi^I \partial \psi^J \partial \psi^K}.
\end{aligned} \qquad (3.81)$$

It is not difficult to show that $\widetilde{\mathcal{D}}^2 = 0$ for any bi-vector;

$$\begin{aligned}
\widetilde{\mathcal{D}}_1^2 &= 0, \\
\widetilde{\mathcal{D}}_1\widetilde{\mathcal{D}}_2 + \widetilde{\mathcal{D}}_2\widetilde{\mathcal{D}}_1 &= 0, \\
\widetilde{\mathcal{D}}_1\widetilde{\mathcal{D}}_3 + \widetilde{\mathcal{D}}_3\widetilde{\mathcal{D}}_1 + \widetilde{\mathcal{D}}_2^2 &= 0, \\
\widetilde{\mathcal{D}}_2\widetilde{\mathcal{D}}_3 + \widetilde{\mathcal{D}}_3\widetilde{\mathcal{D}}_2 &= 0, \\
\widetilde{\mathcal{D}}_3^2 &= 0.
\end{aligned} \qquad (3.82)$$

Thus we obtain another BV_∞ structure.

We encountered the operator $\widetilde{\mathcal{D}}_1$ and $\widetilde{\mathcal{D}}_2$ in Sect. 2.4 where we used the notations Q and $\triangle_\pi$. The latter generated, for $[\pi, \pi]_S = 0$, the covariant Schouten-Nijenhuis bracket on the differential forms. Those operators induced a dGBV structure on the algebra $\mathcal{O}(\Pi TX)$ of functions on ΠTX. We remark that Manin associated a Frobenius structure to a special kind of dGBV algebra [61]. It is shown that the above dGBV algebra defines a Frobenius structure when X is symplectic manifold satisfying the strong Lefschetz condition [62]. Any Kähler manifold is such a manifold, and the above construction reproduces the so called AKS theory [13]. AKS theory is the A model version of Kodaria-Spencer gravity [12].

In general $[\pi, \pi]_S \neq 0$ and we find a homotopy version of a dGBV algebra. It will be interesting to examine the corresponding deformation of the Frobenius structure. We note that the relation between this subsection and Sect. 3.3 can be seen as the relation of cochains versus chains. We remark that there is also a non-commutative version of the above differential operators at least for $h^{IJK} = 0$ [83, 81]. In such a case the cohomology of $\widetilde{\mathcal{D}}$ is closely related to the periodic cyclic homology of the Hochschild chain complex of the deformed algebra of functions on X with the star product. It will be interesting to see if $\widetilde{\mathcal{D}}$ in general has a non-commutative version,

which is not necessarily associative. We also note that the periodic cyclic homology is the non-commutative version of the de Rham cohomology. Can there be an A model on non-commutative space? If so the extended moduli space of the theory may be identified with the periodic cyclic homology.

3.5.2 B boundary conditions

Now we assume that the target space X is a complex Calabi-Yau space. We introduce a complex structure and consider the open membrane theory without background,

$$\boldsymbol{S}_o = \int_N d^3\theta \left(\boldsymbol{\psi}^i D\boldsymbol{\chi}_i + \boldsymbol{\psi}^{\bar{i}} D\boldsymbol{\chi}_{\bar{i}} + \boldsymbol{F}_i D\boldsymbol{u}^i + \boldsymbol{F}_{\bar{i}} D\boldsymbol{u}^{\bar{i}} + \boldsymbol{F}_i \boldsymbol{\psi}^i + \boldsymbol{F}_{\bar{i}} \boldsymbol{\psi}^{\bar{i}} \right).$$
(3.83)

Now consider the original boundary conditions. We can exchange the boundary conditions of $\boldsymbol{\chi}_{\bar{i}}$ and $\boldsymbol{\psi}^{\bar{i}}$, while maintaining the original boundary conditions for $\boldsymbol{\chi}_i$ and $\boldsymbol{\psi}^i$. Then $\chi_{\bar{i}}(x) = \psi^i(x) = 0$ and the 1-forms $H_{\bar{i}}(x)$ and $A^i(x)$ vanish along the direction tangent to $\partial\Sigma$ for $x \in \partial\Sigma$. Then the action functional (3.83) satisfies the master equation. Now the boundary observables are derived from functions on $\Pi\mathcal{T}^*(\Pi\overline{\mathcal{T}}X)$, namely the elements of $\oplus\Omega^{0,\bullet}(X, \wedge^\bullet \mathcal{T}X)$. The general form of the ghost number $U = 2$ function on $\Pi\mathcal{T}^*(\Pi\overline{\mathcal{T}}X)$ is

$$\beta := a + \kappa + b, \qquad \begin{cases} a := a_{\bar{i}}{}^j \chi_j \psi^{\bar{i}}, \\ \kappa := \dfrac{1}{2}\kappa_{\overline{ij}} \psi^{\bar{i}} \psi^{\bar{j}}, \\ b := \dfrac{1}{2}b^{ij} \chi_i \chi_j. \end{cases}$$
(3.84)

We note that the BRST transformation laws restricted to boundary are

$$\begin{aligned} \boldsymbol{Q}_o u^i &= 0, & \boldsymbol{Q}_o \chi_i &= 0, \\ \boldsymbol{Q}_o u^{\bar{i}} &= \psi^{\bar{i}}, & \boldsymbol{Q}_o \psi^{\bar{i}} &= 0. \end{aligned}$$
(3.85)

Hence the BRST charge acts like the $\overline{\partial}$-operators.

The undeformed theory induces the following differential operators acting on functions on $\Pi\mathcal{T}^*(\Pi\overline{\mathcal{T}}X)$ – which can be identified with the elements of $\oplus\Omega^{0,\bullet}(X, \wedge^\bullet \mathcal{T}X)$ –

$$\mathcal{D}_o = \overline{\partial} + \hbar\triangle_T, \qquad \begin{cases} \overline{\partial} = \psi^{\bar{i}} \dfrac{\partial}{\partial u^{\bar{i}}}, \\ \triangle_T = \dfrac{\partial^2}{\partial u^i \partial \chi_i}. \end{cases}$$
(3.86)

We have $\mathcal{D}_o^2 = 0$ and

$$\overline{\partial}^2 = \overline{\partial}\triangle_T + \triangle_T\overline{\partial} = \triangle_T^2 = 0. \tag{3.87}$$

The operators together with the wedge product endow $\oplus\Omega^{0,p}(X, \wedge^q \mathcal{T}X)$ with the structure of dGBV algebra [7, 61, 6]. The operator $\triangle_T$ generates a bracket (Tian-Todorov lemma) $[.,.]_T$, whose holomorphic Schouten-Nijenhuis bracket together with the wedge product on forms define a dGBV algebra. This forms the classical algebra of observables. We note that the solution space of the MC equation of this classical algebra modulo equivalences defines the so-called extended moduli space of complex structures on X [7, 6]. This also induces a Frobenius structure [7, 61], generalizing [12], which is relevant to mirror symmetry.

Now we consider a degree $U = 3$ function γ on $\Pi T(\Pi T^* X)$ satisfying

$$\gamma|_{\psi^i = \chi_{\overline{i}} = F_I = 0} = 0,$$
$$Q_o\gamma + \frac{1}{2}\{\gamma, \gamma\}_P = 0, \tag{3.88}$$

compare with (3.31). Then the action functional

$$\boldsymbol{S}_\gamma = \boldsymbol{S}_o + \int_N d^3\theta \, \boldsymbol{\phi}^*(\gamma) \tag{3.89}$$

satisfies the BV master equation. The deformation term will deform the dGBV algebra (the classical algebra of observables) as a 2-algebra. The deformation is controlled by the MC equation (3.88) of the 3-algebra. It will be interesting to examine the corresponding deformation of the Frobenius structure.

Here we determine a class of solutions of (3.88). The desired γ is given by

$$\gamma = F_i \frac{\partial\beta}{\partial\chi_i} + \psi^i \frac{\partial\beta}{\partial u^i} + \psi^{\overline{i}} \frac{\partial\beta}{\partial u^{\overline{i}}} + \frac{1}{2}[\beta, \beta]_T$$
$$+ \frac{1}{3!}c_{ijk}\left(\psi^i + \frac{\partial\beta}{\partial\chi_i}\right)\left(\psi^j + \frac{\partial\beta}{\partial\chi_j}\right)\left(\psi^k + \frac{\partial\beta}{\partial\chi_k}\right), \tag{3.90}$$

satisfying

$$\overline{\partial}\beta + \frac{1}{2}[\beta, \beta]_T + \frac{1}{3!}[\beta, \beta, \beta]_T = 0,$$
$$\overline{\partial}c + [\beta, c]_T = 0. \tag{3.91}$$

Here we used the following definitions

$$c := \frac{1}{3!} c_{ijk} \psi^i \psi^j \psi^k,$$

$$[\beta, \beta, \beta]_T := c_{ijk} \frac{\partial \beta}{\partial \chi_i} \frac{\partial \beta}{\partial \chi_j} \frac{\partial \beta}{\partial \chi_k}. \tag{3.92}$$

The first condition in (3.91) comes from the first condition in (3.88), while the second condition in (3.91) comes from the second condition in (3.88).

We note that the first equation in (3.91) is the flatness condition $\overline{\partial}_\beta^2 = 0$ of the "covariant" derivative $\overline{\partial}_\beta$;

$$\overline{\partial}_\beta := \overline{\partial} + [\beta, .]_T + \frac{1}{2}[\beta, \beta, .]_T. \tag{3.93}$$

Thus the equations (3.91) become

$$\overline{\partial}_\beta^2 = 0, \qquad \overline{\partial}_\beta c = 0, \tag{3.94}$$

where we used $[\beta, \beta, c]_T = 0$. The equations in (3.91) can be written as follows;

$$\overline{\partial} a + \frac{1}{2}[a, a]_T = 0, \qquad \overline{\partial}\kappa + [a, \kappa]_T + \frac{1}{3!}[a, a, a]_T = 0,$$

$$[b, \kappa]_T + \frac{1}{2}[b, a, a]_T = 0, \qquad \overline{\partial} b + [a, b]_T + \frac{1}{2}[a, b, b]_T = 0, \tag{3.95}$$

$$\frac{1}{2}[b, b]_T + \frac{1}{3!}[b, b, b]_T = 0, \qquad \overline{\partial} c + [a, c]_T = 0.$$

Now the deformed theory induces the following differential operators acting on functions on $\Pi \mathcal{T}^*(\Pi \overline{\mathcal{T}} X)$, identified with the elements of $\oplus \Omega^{0,\bullet}(X, \wedge^\bullet \mathcal{T} X)$;

$$\mathcal{D}_\gamma = \mathcal{D}_1 + \hbar \mathcal{D}_2 + \hbar \mathcal{D}_3, \tag{3.96}$$

where

$$\mathcal{D}_1 = \psi^{\overline{i}} \frac{\partial}{\partial u^{\overline{i}}} + \frac{\partial \beta}{\partial \chi_i} \frac{\partial}{\partial u^i} + \frac{\partial \beta}{\partial u^i} \frac{\partial}{\partial \chi_i} + \frac{1}{2} c_{ijk} \frac{\partial \beta}{\partial \chi_i} \frac{\partial \beta}{\partial \chi_j} \frac{\partial}{\partial \chi_k},$$

$$\mathcal{D}_2 = \frac{\partial^2}{\partial u^i \partial \chi_i} + \frac{1}{2} c_{ijk} \frac{\partial \beta}{\partial \chi_i} \frac{\partial^2}{\partial \chi_j \partial \chi_k}, \tag{3.97}$$

$$\mathcal{D}_3 = c_{ijk} \frac{\partial^3}{\partial \chi_i \partial \chi_j \partial \chi_k}.$$

Once again we have $\mathcal{D}_\gamma^2 = 0$ from (3.91);

$$\mathcal{D}_1^2 = 0,$$
$$\mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2\mathcal{D}_1 = 0,$$
$$\mathcal{D}_1\mathcal{D}_3 + \mathcal{D}_3\mathcal{D}_1 + \mathcal{D}_2^2 = 0, \quad (3.98)$$
$$\mathcal{D}_2\mathcal{D}_3 + \mathcal{D}_3\mathcal{D}_2 = 0,$$
$$\mathcal{D}_3^2 = 0.$$

The operators together with the wedge product endow $\oplus\Omega^{0,p}(X, \wedge^q \mathcal{T}X)$ with a structure of BV_∞ algebra.

We may consider more general boundary observables of the theory than β in (3.84). For this we consider an arbitrary function $\alpha(u^i, u^{\bar{i}}, \chi_i, \psi^{\bar{i}})$ of $(u^I, \chi_i, \psi^{\bar{i}})$ on $\Pi\mathcal{T}^*\Pi(\overline{\mathcal{T}}X)$, which is an element of $\oplus\Omega^{0,\bullet}(X, \wedge^\bullet \mathcal{T}X)$. Now we can replace β with α in the equations (3.90) and (3.91) to obtain a more general solution of the BV master equation (3.88). We can also replace β with α in (3.97). Then the BV master equation is given by

$$\overline{\partial}\alpha + \frac{1}{2}[\alpha, \alpha]_T + \frac{1}{3!}[\alpha, \alpha, \alpha]_T = 0,$$
$$\overline{\partial}c + [\alpha, c]_T = 0, \quad (3.99)$$

while the differentials acting on functions on $\Pi\mathcal{T}^*(\Pi\overline{\mathcal{T}}X)$ are

$$\mathcal{D}_1 = \psi^{\bar{i}}\frac{\partial}{\partial u^{\bar{i}}} + \frac{\partial\alpha}{\partial\chi_i}\frac{\partial}{\partial u^i} + \frac{\partial\alpha}{\partial u^i}\frac{\partial}{\partial\chi_i} + \frac{1}{2}c_{ijk}\frac{\partial\alpha}{\partial\chi_i}\frac{\partial\alpha}{\partial\chi_j}\frac{\partial}{\partial\chi_k},$$
$$\mathcal{D}_2 = \frac{\partial^2}{\partial u^i\partial\chi_i} + \frac{1}{2}c_{ijk}\frac{\partial\alpha}{\partial\chi_i}\frac{\partial^2}{\partial\chi_j\partial\chi_k}, \quad (3.100)$$
$$\mathcal{D}_3 = c_{ijk}\frac{\partial^3}{\partial\chi_i\partial\chi_j\partial\chi_k}.$$

Now we consider the special case that $c = 0$. Then we are left with only the boundary degrees of freedom and the master equation (3.88) or equivalently (3.99) reduces to

$$\overline{\partial}\alpha + \frac{1}{2}[\alpha, \alpha]_T = 0. \quad (3.101)$$

The differential operators $\mathcal{D}$ acting on functions on $\Pi\mathcal{T}^*(\Pi\overline{\mathcal{T}}X)$ become

$$\mathcal{D}_1 = \psi^{\bar{i}}\frac{\partial}{\partial u^{\bar{i}}} + \frac{\partial\alpha}{\partial\chi_i}\frac{\partial}{\partial u^i} + \frac{\partial\alpha}{\partial u^i}\frac{\partial}{\partial\chi_i},$$
$$\mathcal{D}_2 = \triangle_T = \frac{\partial^2}{\partial u^i\partial\chi_i}, \quad (3.102)$$
$$\mathcal{D}_3 = 0.$$

We see that the boundary deformations correspond to deformations of the first order differential operator. We shall see that those correspond to deformations of the BRST operator of the B model of the topological closed string theory. We shall also see that the solution space of (3.101) modulo equivalences defines the extended moduli space of the B model of the topological string theory or the extended moduli space of complex structures. Now turning on the c-field gives rise to bulk deformations beyond the extended moduli space of the boundary theory.

4 Back to the strings

In this section we discuss some applications to homological mirror symmetry. In the previous section we already discussed about related issues, though on a level one step up (the open membrane).

4.1 Relation with $N_{ws} = (2,2)$ supersymmetric sigma models

We consider the following two functions carrying $U = 3$

$$
\begin{aligned}
h &= F_I \psi^I, \\
\widetilde{h} &= b^{IJ} F_I \chi_J + \frac{1}{2} \partial_K b^{IJ} \psi^k \chi_I \chi_J,
\end{aligned}
\tag{4.1}
$$

where $\pi = \frac{1}{2} b^{IJ} \chi_I \chi_J$ is a Poisson bivector, i.e. $[\pi, \pi]_S = 0$. Then it is easy to check that

$$
\{h, h\}_P = 0, \qquad \{h, \widetilde{h}\}_P = 0, \qquad \{\widetilde{h}, \widetilde{h}\}_P = 0.
\tag{4.2}
$$

Thus the Hamiltonian vectors $\{h....\}_P = Q_o$ and $\{\widetilde{h}, ...\}_P = \widetilde{Q}_o$ satisfy the following relations

$$
\{Q, Q\} = 0, \qquad \{Q, \widetilde{Q}\} = 0, \qquad \{\widetilde{Q}, \widetilde{Q}\} = 0.
\tag{4.3}
$$

Explicitly

$$
\begin{aligned}
Q_o =& \psi^I \frac{\partial}{\partial u^I} + F_I \frac{\partial}{\partial \chi_I}, \\
\widetilde{Q}_o =& b^{IJ} \chi_J \frac{\partial}{\partial u^I} + \frac{1}{2} \partial_I b^{JK} \chi_J \chi_K \frac{\partial}{\partial \chi_I} - \left(b^{IJ} F_J + \partial_K b^{IJ} \psi^k \chi_J \right) \frac{\partial}{\partial \psi^I} \\
& - \left(\partial_I b^{JK} F_J \chi_K + \frac{1}{2} \partial_I \partial_J b^{LM} \psi^J \chi_L \chi_M \right) \frac{\partial}{\partial F^I}.
\end{aligned}
\tag{4.4}
$$

We note that the two odd vectors Q_o and $\widetilde{Q}_o$ are the supercharges of a two dimensional $N_{ws} = (1,1)$ supersymmetric non-linear sigma model after dimensional reductions to zero dimension. Then the commuting coordinate field F_I correspond to the auxiliary fields. Now we assume that X is a Kähler manifold and the bivector is the inverse of the Kähler form. We introduce a complex structure and decompose the coordinates according to $u^I = u^i + u^{\bar{i}}$. We decompose the tangent vectors, $\psi^I = \psi^i + \psi^{\bar{i}}$, and cotangent vectors, $\chi_I = \chi_{\bar{i}} + \chi_i$ and $F_I = F_{\bar{i}} + F_i$, accordingly. Then the odd vectors Q_o and $\widetilde{Q}_o$ are decomposed into holomorphic and anti-holomorphic vectors; $Q_o = Q'_o + Q''_o$, $\widetilde{Q}_o = \widetilde{Q}'_o + \widetilde{Q}''_o$ and all four odd vectors are mutually nilpotent. Again the four odd vectors correspond to the supercharges of a two dimensional $N_{ws} = (2,2)$ supersymmetric non-linear sigma model after dimensional reduction to zero dimensions. Now we imagine undoing the dimensional reduction. Then our four odd vectors transform as the left or right moving spinors under the two-dimensional rotation group of a Riemann surface Σ. They also carry so called $U(1)_{\mathcal{R}}$ charges.

As was originally proposed [21, 57], the mirror symmetry is very natural from a physical viewpoint. On the other hand the symmetry becomes quickly mystifying once we translate it into mathematical language in terms of Gromov-Witten invariants and the variation of Hodge structures. An application of mirror symmetry in adopting the latter viewpoint marked the first grand success of the mirror conjecture in algebra-arithmetic geometry [16], after the first construction of a mirror pair [35]. Later Witten proposed a unified and effective viewpoint based on twisted versions (A and B models) of $\mathcal{N} = (2,2)$ supersymmetric sigma models [92]. Additionally, Witten also argued that both the original A and B models should be extended and the mirror symmetry would be more obvious in the extended models. In this language the mirror symmetry is the physical equivalence between the extended A model $A^e(X)$ on X and the extended B model $B^e(Y)$ on the mirror Y.

The twisting procedure is explained in detail in [92, 12]. There are two different twistings, leading to the A and the B model, respectively. There is also a half-twisted model which we will not consider here. Note that the B model makes sense iff X is a Calabi-Yau space with Ricci-flat metric. After twisting two supercharges transform as scalars on Σ. The other two supercharges transform as vectors. The auxiliary fields F_i and $F_{\bar{i}}$ transform as a vector for the A model, while decomposing into a scalar and a two-form in the B model. For the A model, the two anti-commuting scalar

supercharges (BRST charges) are given by

$$\mathcal{Q}'_o = \psi^i \frac{\partial}{\partial u^i} + \dots,$$
$$\mathcal{Q}''_o = \psi^{\bar{i}} \frac{\partial}{\partial u^{\bar{i}}} + \dots,$$

(4.5)

where the omitted the part involving non-scalar fields. For the B model the two BRST charges are given by

$$\mathcal{Q}''_o = \psi^{\bar{i}} \frac{\partial}{\partial u^{\bar{i}}} + F_i \frac{\partial}{\partial \chi_i} + \dots,$$
$$\widetilde{\mathcal{Q}}''_o = b^{\bar{i}j} \chi_j \frac{\partial}{\partial u^i} - \left(b^{\bar{i}j} F_j + \partial_{\bar{k}} b^{\bar{i}j} \psi^k \chi_j \right) \frac{\partial}{\partial \psi^{\bar{i}}} + \dots,$$

(4.6)

where we again omitted the part involving non-scalar fields.

At this level we already see the sharp distinction between the A and the B model. As a general principle observables of the theory are constructed by BRST cohomology classes. The zero-dimensional observables, transforming as scalars, are constructed from scalar fields. For the A model the answer is obvious. The two BRST charges originated from the ∂ and $\overline{\partial}$ differentials of the target space X. Thus the zero-dimensional observables are the pull-backs of de Rham cohomology classes of X after a parity change.

On the other hand the $\mathcal{Q}''_o$ charge can better be interpreted as $\overline{\partial}$-operator on the total superspace $\Pi T^* X$. Furthermore, the ∂ part of BRST charge does not exist. The other BRST charge $\widetilde{\mathcal{Q}}''_o$ can be interpreted as the holomorphic part of the Poisson differential on the superspace $\Pi T^* X$. Consequently the zero-dimensional observables of the B model should come from the Dolbeault cohomology of the superspace $\Pi T^* X$. However it is not clear how it can be defined. The usual approach is based on an on-shell formalism[14] and taking the diagonal. The zero-dimensional observables are elements of the sheaf cohomology $H^{0,\bullet}(X, \wedge^\bullet T^{1,0} X)$, modulo equations of motion. Then one solves the descent equations to construct two dimensional observables. However some problems appear in constructing higher dimensional observables by solving the descent equations. Two dimensional observables are very important since one can add them by integrating over Σ to the action functional to define a family of the theory. Witten gave a recipe for the B model with some first order analysis and shows that the classical moduli space $\mathcal{M}_{cl}$ of complex structures should be extended [92]. He argued that the tangent space of the extended moduli space $\mathcal{M}$ at a

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[14]On-shell the term $(b^{\bar{i}j} F_j + \partial_{\bar{k}} b^{\bar{i}j} \psi^k \chi_j)$ vanishes due to an equation of motion.

classical point is $\oplus H^{0,\bullet}(X, \wedge^\bullet T^{1,0}X)$. A noble point of the above analysis is that the BRST transformation laws should be changed recursively. This is a perfect lead to the method of BV quantization.

The question on the extended moduli space $\mathcal{M}$ was the starting point of the homological mirror conjecture of Kontsevich [49]. Roughly there are three questions. (i) define (or find equations for) the moduli space $\mathcal{M}$. (ii) define generalized periods. (iii) what is the meaning of $\mathcal{M}$ or which kind of object does it parameterize?. The first and second questions are answered by Barannikov and Kontsevich [7, 6] using purely mathematical techniques based on modern deformation theory. They also constructed explicit formulas for the tree level potential, which is the generalization of the Kodaira-Spensor theory of Bershadsky et. al [12] (see also [61]). However the corresponding generalization for string loops is not yet achieved. The third question remains still mysterious. Kontsevich conjectured that $\mathcal{M}$ parameterizes the A_∞ deformation of $D^b(Coh(X))$. This conjecture was answered affirmatively in [6], based on the formality theorem of Kontsevich.

The homological mirror conjecture seems to be closely related to the SYZ conjecture [78, 84, 53]. In many respects the SYZ construction makes mirror symmetry obvious as a physical equivalence [78]. Recently Hori et. al. [39, 40] developed another physically natural approach which has a highly constructive power.

4.1.1 Cohomological ABC model

It is possible to describe the A, B and Catteneo-Felder (C in short) models in a unified fashion. For this we consider a theory based on the following map

$$\varphi : \Pi T\Sigma \to \Pi T^*(\Pi(TX)). \tag{4.7}$$

We let $\mathcal{A}_\Sigma$ denotes the space of all maps. We denote a system of local coordinates on ΠTX by $(\{u\}^I, \{\psi^I\})$, while by $(\{\chi_I\}, \{F_I\})$ for the fiber of $\Pi T^*(TX)$. We assign the ghost numbers of the coordinates by $U(u^I, \psi_I, \chi_I, F_I) = (0, 1, 1, 0)$. We may use the same component notations as for the membrane case (3.18) by simply setting the 3-form parts to zero. We describe a map (4.7) by the superfields

$$(\boldsymbol{u}^I, \boldsymbol{\psi}^I, \boldsymbol{\chi}_I, \boldsymbol{F}_I) := (u^I, \chi_I, \psi^I, F_I)(x^\mu, \theta^\mu), \tag{4.8}$$

where $(\{x^\mu\}, \{\theta^\mu\})$, $\mu = 1, 2$ denote a set of local coordinates on $\Pi T\Sigma$. On $\Pi T^*\Pi(TX)$ we have the following odd symplectic form with $U = 1$,

$$\omega_\Sigma = du^I d\chi_I + d\psi^I dF_I. \tag{4.9}$$

The corresponding odd Poisson bracket is an extension of the Schouten-Nijenhuis bracket. On $\mathcal{A}_\Sigma$ we have an induced odd symplectic form $\boldsymbol{\omega}_\Sigma$ with $U = -1$

$$\boldsymbol{\omega}_\Sigma = \int_\Sigma d^2\theta \left(\delta \boldsymbol{u}^I \delta \boldsymbol{\chi}_I + \delta \boldsymbol{\psi}^I \delta \boldsymbol{F}_I \right). \tag{4.10}$$

Thus the BV bracket of the theory originates form the extended Schouten-Nijenhuis bracket. Now we may follow the usual steps, as in the previous sections.

We start from the following action functional,

$$\boldsymbol{S}_o = \int_\Sigma d^2\theta \left(\boldsymbol{\chi}_I D\boldsymbol{u}^I + \boldsymbol{F}_I D\boldsymbol{\psi}^I + \boldsymbol{\chi}_I \boldsymbol{\psi}^I \right). \tag{4.11}$$

The above action functional satisfies the BV master equation. The BRST transformation laws can be obtained by the odd Hamiltonian vector $\boldsymbol{\mathcal{Q}}_o$ of $\boldsymbol{S}_o$;

$$\boldsymbol{\mathcal{Q}}_o = \left(D\boldsymbol{u}^I + \boldsymbol{\psi}^I \right) \frac{\partial}{\partial \boldsymbol{u}^I} + D\boldsymbol{\psi}^I \frac{\partial}{\partial \boldsymbol{u}^I} + \left(D\boldsymbol{F}_I + \boldsymbol{\chi}_I \right) \frac{\partial}{\partial \boldsymbol{F}_I} + D\boldsymbol{\chi}_I \frac{\partial}{\partial \boldsymbol{\chi}_I}. \tag{4.12}$$

Now we consider the following general deformation preserving the ghost number

$$\boldsymbol{S}_\Gamma = \boldsymbol{S}_o + \int_\Sigma d^2\theta \boldsymbol{\varphi}^*(\Gamma)$$
$$= \boldsymbol{S}_o + \int_\Sigma d^2\theta \left(a_I{}^J(\boldsymbol{u}^L) \boldsymbol{\chi}_J \boldsymbol{\psi}^I + \frac{1}{2} \kappa_{IJ}(\boldsymbol{u}^L) \boldsymbol{\psi}^I \boldsymbol{\psi}^J + \frac{1}{2} b^{IJ}(\boldsymbol{u}^L) \boldsymbol{\chi}_I \boldsymbol{\chi}_J \right), \tag{4.13}$$

where $\Gamma = A + K + \pi \in \Omega^2(X) \oplus \Omega^1(X, TX) \oplus \Omega^0(X, \wedge^2 TX)$ with suitable parity changes;

$$A = a_I{}^J(u^L) \chi_I \psi^J,$$
$$K = \frac{1}{2} \kappa_{IJ}(u^L) \psi^I \psi^J, \tag{4.14}$$
$$\pi = \frac{1}{2} b^{IJ}(u^L) \chi_I \chi_J.$$

The above deformation terms do not have any dependence on $\boldsymbol{F}_I$, which simplifies our analysis a lots.[15] The above action functional satisfies the BV

[15] Hence we are using $\boldsymbol{F}_I$ as auxiliary devices.

master equation if and only if

$$dK + [A, K]_S = 0,$$
$$dA + \frac{1}{2}[A, A]_S + [\pi, K]_S = 0,$$
$$d\pi + [A, \pi]_S = 0, \tag{4.15}$$
$$[\pi, \pi]_S = 0,$$

where $d := \psi^I \frac{\partial}{\partial u^I}$ is the (parity changed) exterior derivative on X and the bracket above denotes the Schouten-Nijenhuis bracket together with the wedge product on forms. The above equation can be summarized into a single equation,

$$d\Gamma + \frac{1}{2}[\Gamma, \Gamma]_S = 0. \tag{4.16}$$

It seems reasonable to relate the moduli space defined by the above equations to the moduli space of $N = (2, 2)$ superconformal field theory in two-dimensions.

Now we consider the special cases of $\boldsymbol{S}_\gamma$. The C model[16] can be obtained by setting $\boldsymbol{\psi}^I = \boldsymbol{F}_I = 0$ for all I

$$\boldsymbol{S}^C = \int_\Sigma d^2\theta \left(\boldsymbol{\chi}_I D\boldsymbol{u}^I + \frac{1}{2}\boldsymbol{b}^{IJ}\boldsymbol{\chi}_I\boldsymbol{\chi}_J \right). \tag{4.17}$$

The A model in Sect. 2.4. can be obtained by setting $\boldsymbol{\chi}_I = \boldsymbol{F}_I = 0$ for all I;

$$\boldsymbol{S}^A = \frac{1}{2} \int_\Sigma d^2\theta \left(\boldsymbol{\kappa}_{IJ}\boldsymbol{\psi}^I\boldsymbol{\psi}^J \right). \tag{4.18}$$

We may also consider the open string version of the A model [93, 27, 25, 28]. It is worth mentioning that the BV master equation does not tell anything on the possible boundary conditions, since the master action functional does not have a kinetic term before gauge fixing.

Next we turn to the B model.

4.2 Extended B model

The B model is more complicated. First of all X must be a Calabi-Yau manifold. We pick a complex structure on X and $\boldsymbol{\psi}^i = \boldsymbol{F}_{\bar{\imath}} = 0$ for all i.

---

[16]We note that the relation between the C model and the ABC model is analogous to the relation between cohomological field theory and balanced cohomological field theory [22].

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Then $\boldsymbol{S}_\gamma$ in (4.16) becomes

$$\boldsymbol{S}^B_\beta = \boldsymbol{S}^B_o + \int_\Sigma \boldsymbol{\varphi}^*(\beta), \tag{4.19}$$

where

$$\boldsymbol{S}^B_o = \int_\Sigma d^2\theta \left(\boldsymbol{\chi}_i D\boldsymbol{u}^i + \boldsymbol{\chi}_{\bar{i}} D\boldsymbol{u}^{\bar{i}} + \boldsymbol{\chi}_{\bar{i}}\boldsymbol{\psi}^{\bar{i}} \right), \tag{4.20}$$

and

$$\int_\Sigma \boldsymbol{\varphi}^*(\beta) = \int_\Sigma d^2\theta \left(\boldsymbol{a}_{\bar{i}}^j \boldsymbol{\chi}_j \boldsymbol{\psi}^{\bar{i}} + \frac{1}{2}\boldsymbol{\kappa}_{\overline{ij}}\boldsymbol{\psi}^{\bar{i}}\boldsymbol{\psi}^{\bar{i}} + \frac{1}{2}\boldsymbol{b}^{ij}\boldsymbol{\chi}_i\boldsymbol{\chi}_j \right). \tag{4.21}$$

The authors of [2] showed that the standard B model action functional can be obtained by a suitable gauge fixing of the master action functional $\boldsymbol{S}^B_o$ given by (4.20).[17] For us it suffices to mention that H_i, which is the 1-form component of $\boldsymbol{\chi}_i$, is replaced by

$$H_i = g_{i\bar{i}} * du^{\bar{i}} + \ldots. \tag{4.22}$$

We will use this later. We also take the only non-vanishing components of the superfields $\boldsymbol{\chi}_{\bar{i}}$ to be the two-form parts. Then the kinetic term $\int_\Sigma d^2\theta\, \boldsymbol{\chi}_{\bar{i}} D\boldsymbol{u}^{\bar{i}}$ in (4.20) vanishes. Thus the undeformed master action functional for the B model is actually given by

$$\boldsymbol{S}^B_o = \int_\Sigma d^2\theta \left(\boldsymbol{\chi}_i D\boldsymbol{u}^i + \boldsymbol{\chi}_{\bar{i}}\boldsymbol{\psi}^{\bar{i}} \right). \tag{4.23}$$

We note that the conditions $\boldsymbol{\psi}^i = \boldsymbol{F}_{\bar{i}} = \chi_{\bar{i}} = H_{\bar{i}} = 0$ are equivalent to the B boundary conditions for the open membrane.

Now we regard the terms in (4.21) as deformations of the theory preserving the ghost number symmetry. The BV master equation reduces to

$$\overline{\partial}\beta + \frac{1}{2}[\beta, \beta]_T = 0, \tag{4.24}$$

where $\beta = \kappa + a + b \in \Omega^{0,2}(X) \oplus \Omega^{0,1}(X, \mathcal{T}X) \oplus \Omega^{0,0}(X, \wedge^2\mathcal{T}X)$. Note that β is defined as (3.84). The bracket denotes a holomorphic version of the Schouten-Nijenhuis bracket and we have the wedge product on forms. By the Tian-Todorov lemma the bracket can be obtained from the holomorphic top-form on X, etc. [12, 7].

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[17] They actually start from the action functional $\boldsymbol{S}_o$ in (4.11). They regard the conditions $\boldsymbol{\psi}^i = \boldsymbol{F}_{\bar{i}} = 0$ as part of the gauge fixing.

We may consider a more general deformation of the theory (4.20). A zero-dimensional observable is a function of the scalar components of the superfields (4.8). For this we consider an arbitrary function $\alpha(u^i, u^{\bar{i}}, \chi_i, \psi^{\bar{i}})$ of $(u^I, \chi_i, \psi^{\bar{i}})$ on $\Pi\mathcal{T}^*\Pi(\overline{\mathcal{T}}X)$, which is an element of $\oplus\Omega^{0,\bullet}(X, \wedge^\bullet \mathcal{T}X)$. Then $\alpha(x^\mu)$ is a BV BRST observable if and only if $\boldsymbol{Q}_o\alpha = \psi^{\bar{i}}\partial_{\bar{i}}\alpha = 0$. Thus $\boldsymbol{Q}_o$ acts on α as the $\overline{\partial}$ operators on X; α is an element of $\oplus H^{0,\bullet}(X, \wedge^\bullet \mathcal{T}X)$. We consider a basis $\{\alpha_a\}$ of $\oplus H^{0,\bullet}(X, \wedge^\bullet \mathcal{T}X)$. Then we have the following family of B models

$$\boldsymbol{S}^B(\{t^a\}) = \boldsymbol{S}^B_o + \sum t^a \int_\Sigma d^2\theta \, \boldsymbol{\varphi}^*(\alpha_a). \tag{4.25}$$

The above action functional satisfies the BV master equation if and only if

$$\overline{\partial}(t^a\alpha_a) + \frac{1}{2}\left[\sum t^a\alpha_a, \sum t^b\alpha_b\right]_T = 0. \tag{4.26}$$

It follows that the BV master equation holds up to first order in t^a. To go beyond the first order we consider a certain function $\alpha(t)$, which is an element of $\oplus\Omega^{0,\bullet}(X, \wedge^\bullet \mathcal{T}X)$, having the the following formal expansion:

$$\alpha(t) = t^a\alpha_a + \sum_{n>1} t^{a_1}\dots t^{a_n}\alpha_{a_1\dots a_n}. \tag{4.27}$$

Then we consider the family of action functionals $\boldsymbol{S}(t)$ defined by

$$\boldsymbol{S}^B(t) = \boldsymbol{S}^B_o + \int_\Sigma d^2\theta \, \boldsymbol{\varphi}^*(\alpha(t)). \tag{4.28}$$

The above deformation is said to be well-defined if $\boldsymbol{S}(t)$ satisfies the BV master equation

$$\left(\boldsymbol{S}^B(t), \boldsymbol{S}^B(t)\right)_{BV} = 0. \tag{4.29}$$

Equivalently,

$$\boldsymbol{Q}_o \int_\Sigma d^2\theta \, \boldsymbol{\varphi}^*(\alpha(t)) + \frac{1}{2}\left(\int_\Sigma d^2\theta \, \boldsymbol{\varphi}^*(\alpha(t)), \int_\Sigma d^2\theta \, \boldsymbol{\varphi}^*(\alpha(t))\right)_{BV} = 0. \tag{4.30}$$

The above condition is equivalent to the following MC equation,

$$\overline{\partial}\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)]_T = 0. \tag{4.31}$$

Thus the success of BV quantization is equivalent to having solutions to the above MC equation in the form (4.27). We note that the BV BRST transformation laws for the deformed action functional (4.28) are

$$
\begin{aligned}
\boldsymbol{Q}(t)\boldsymbol{u}^i &= D\boldsymbol{u}^i + \frac{\partial\boldsymbol{\alpha}(t)}{\partial\boldsymbol{\chi}_i}, \\
\boldsymbol{Q}(t)\boldsymbol{u}^{\bar{i}} &= \boldsymbol{\psi}^{\bar{i}}, \\
\boldsymbol{Q}(t)\boldsymbol{\psi}^{\bar{i}} &= 0, \\
\boldsymbol{Q}(t)\boldsymbol{\chi}_i &= D\boldsymbol{\chi}_i + \frac{\partial\boldsymbol{\alpha}(t)}{\partial\boldsymbol{u}^i}, \\
\boldsymbol{Q}(t)\boldsymbol{\chi}_{\bar{i}} &= \frac{\partial\boldsymbol{\alpha}(t)}{\partial\boldsymbol{u}^{\bar{i}}}.
\end{aligned}
\tag{4.32}
$$

The condition that $\boldsymbol{Q}(t)^2 = 0$ is equivalent to the master equation (4.31). The MC equation or master equation (4.31) is precisely the equation for the extended moduli space of complex structures on X, defined by [7]. Here we derived the result as the consistency condition for quantization, completing the original analysis of Witten in [92].

We also obtained a family of B models, which is actually parametrized by the extended moduli space. We call the resulting theory the extended B model. We note that the deformations of the B model leading to the extended B model are deformations of the differential or the BRST charge of the theory. On the other hand the topological open membrane theory with B boundary conditions in Sect. 3.5.2 deforms the bracket as well by the bulk deformations. We also showed that the topological open membrane theory with B boundary conditions contains all the information of the extended B model of topological closed string theory.

Now we turn to the open string version of the extended B model.

4.2.1 Towards physical descriptions of A_∞-deformations

of $D^b Coh(X)$

According to a conjecture of Kontsevich the extended moduli space $\mathcal{M}$ of complex structures parameterizes the A_∞ deformations of $D^b(Coh(X))$ [49, 6]. In this section we sketch a program constructing $D^b(Coh(X))$ and its A_∞-deformations by path integral methods. For the A model side of the story we refer to [28] for a state of art construction. For this we consider the theory on a disc.

We begin by comparing the B model with the C model in Sect. 2. The undeformed C model is given by

$$\boldsymbol{S}_o^C = \int_\Sigma d^2\theta \left(\boldsymbol{\chi}_I D \boldsymbol{u}^I \right) \qquad (4.33)$$

while the undeformed B model is given by

$$\boldsymbol{S}_o^B = \int_\Sigma d^2\theta \left(\boldsymbol{\chi}_i D \boldsymbol{u}^i + \boldsymbol{\chi}_{\bar{i}} \psi^{\bar{i}} \right). \qquad (4.34)$$

In both cases the boundary condition, in order to satisfy the BV master equation, should be that $\boldsymbol{\chi}_I(x)$ vanish along the direction tangent to $\partial\Sigma$ for $x \in \partial\Sigma$. We note that the B model is just the holomorphic version of the C-model but with one additional term compared to the B model. Recall that the boundary observables of C model are given by arbitrary functions on X. The algebra $\mathcal{O}(X)$ of functions (cohomological 1-algebra of X) on X is determined only by the commutative and associative product. As we explained earlier, the undeformed action functional has only the kinetic term, accordingly.

For the B model above a boundary observable is given by a function on $\Pi\overline{\mathcal{T}}X$, i.e., $\oplus\Omega^{0,\bullet}(X)$. The algebra $\mathcal{O}(\Pi\overline{\mathcal{T}}X)$ of functions on $\Pi\overline{\mathcal{T}}X$ has a supercommutative, an associative product, and a differential. The undeformed B model action functional tells us that the differential is the $\overline{\partial}$ operator. This can be seen by the BRST transformation laws restricted to the boundary;

$$\begin{aligned} \boldsymbol{Q}_o \boldsymbol{u}^i &= D\boldsymbol{u}^i, \\ \boldsymbol{Q}_o \boldsymbol{u}^{\bar{i}} &= \psi^{\bar{i}}, \\ \boldsymbol{Q}_o \psi^{\bar{i}} &= 0. \end{aligned} \qquad (4.35)$$

Then $\boldsymbol{Q}_o u^i = 0$, $\boldsymbol{Q}_o u^{\bar{i}} = \psi^{\bar{i}}$ and $\boldsymbol{Q}_o \psi^{\bar{i}} = 0$, which defines the $\overline{\partial}$ operator. Thus the algebra of classical observables is $\mathcal{O}(\Pi\overline{\mathcal{T}}X)$, with a supercommutative and associative product (wedge product) and a differential $\overline{\partial}$, which equip $\mathcal{O}(\Pi\overline{\mathcal{T}}X)$ with a structure of dgLa without bracket.

Now we turn on a deformation preserving the ghost number symmetry. For the C model we have

$$\delta\boldsymbol{S}^C = \int_\Sigma d^2\theta \, \boldsymbol{\varphi}^*(\pi), \qquad (4.36)$$

where $\pi = \frac{1}{2}b^{IJ}\chi_I\chi_J$ denotes a bi-vector on X. The bivector is an element of the 2nd Hochschild cohomology of $\mathcal{O}(X)$, i.e. $\pi \in H^2(\texttt{Hoch}(\mathcal{O}(X)))$.

Now the bulk deformation above deforms the product in a non-commutative direction. The master equation

$$[\pi, \pi]_S = 0, \tag{4.37}$$

implies that the deformed product is associative.

For the B model we have

$$\delta \boldsymbol{S}^B = \int_\Sigma d^2\theta \, \boldsymbol{\varphi}^*(a + b), \tag{4.38}$$

where

$$a = a_{\overline{i}}{}^j \chi_j \psi^{\overline{i}},$$
$$b = \frac{1}{2} b^{ij} \chi_i \chi_j. \tag{4.39}$$

Note that the deformation by $\frac{1}{2} \kappa_{\overline{ij}} \psi^{\overline{i}} \psi^{\overline{j}}$ in (4.21) is not allowed due to the master equation. As before it is not a bulk term. The two objects above correspond to elements of the 2nd Hochschild cohomology of $\mathcal{O}(\Pi \overline{\mathcal{T}} X)$, i.e. $a, b \in H^2(\texttt{Hoch}(\mathcal{O}(\overline{\mathcal{T}} X)))$. According to the mathematical definition $\frac{1}{2} \kappa_{\overline{ij}} \psi^{\overline{i}} \psi^{\overline{j}} \in \Omega^{0,2}(X)$ also belongs to $H^2(\texttt{Hoch}(\mathcal{O}(\Pi \mathcal{T} X)))$ [49]. However it is physically unnatural. In general we identify $H^*(\texttt{Hoch}(\mathcal{O}(\Pi \overline{\mathcal{T}} X)))$ with $\oplus_{p \geq 1, q} \Omega^{0,q}(X, \wedge^p \overline{\mathcal{T}} X)$. The meaning of a deformation by a is obvious; it is a deformation of the complex structure. Comparing with the C model we see that the b term in the bulk generates a non-commutative deformation of the product. The master equation is

$$\overline{\partial} a + \frac{1}{2} [a, a]_T = 0,$$
$$\overline{\partial} b + [a, b]_T = 0, \tag{4.40}$$
$$[b, b]_T = 0.$$

The first equation implies that the deformation of the complex structure is integrable. The last equation implies that the non-commutative deformation is associative. The middle equation may be viewed as a compatibility of the two deformations. The mirror picture for Abelian varieties is discussed in [26] (see also [6] for a non-commutative version of the variation of Hodge structures more for closed string case).

Note that the deformation by a changes the BRST transformation laws restricted to the boundary as follows

$$\boldsymbol{Q} \boldsymbol{u}^i = D \boldsymbol{u}^i + \boldsymbol{a}_i{}^{\overline{j}} \boldsymbol{\psi}_{\overline{j}},$$
$$\boldsymbol{Q} \boldsymbol{u}^{\overline{i}} = \boldsymbol{\psi}^{\overline{i}}, \tag{4.41}$$
$$\boldsymbol{Q} \boldsymbol{\psi}^{\overline{i}} = 0.$$

Thus the boundary observables should be changed accordingly. In the above case it is just the change of complex structure. We may consider the general deformation

$$\delta \boldsymbol{S}^B = \int_\Sigma d^2\theta \boldsymbol{\varphi}^*(\gamma(u^i, u^{\bar{i}}, \chi_i, \psi^{\bar{i}}) \tag{4.42}$$

where $\gamma \in \oplus_{p \geq 1, q} \Omega^{0,q}(X, \wedge^p \mathcal{T}X) \equiv H^*(\texttt{Hoch}(\mathcal{O}(\Pi \overline{\mathcal{T}}X)))$. Then the boundary BRST transformation laws are changed as follows

$$\begin{aligned}
\boldsymbol{Q}u^i &= D\boldsymbol{u}^i + \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\chi}_i}|_{\chi_i = 0}, \\
\boldsymbol{Q}u^{\bar{i}} &= \boldsymbol{\psi}^{\bar{i}}, \\
\boldsymbol{Q}\psi^{\bar{i}} &= 0.
\end{aligned} \tag{4.43}$$

Thus only the element in $\oplus_p \Omega^{0,p}(X, \mathcal{T}X)$ affects the boundary BRST transformation laws. Such elements correspond to the classical deformation of complex structures. Thus, without loss of generality, we say that the boundary observables are determined by the $\overline{\partial}$ cohomology among functions on $\Pi \overline{\mathcal{T}}X$.

Kontsevich's formality theorem implies that the dgLa on $\oplus_{p,q} \Omega^{0,q}(X, \wedge^p \mathcal{T}X)$ or, equivalently, on $H^*(\texttt{Hoch}(\mathcal{O}(\Pi \overline{\mathcal{T}}X)))$ is quasi-isomorphic to the Lie algebra of local Hochschild cochains $\texttt{Hoch}(\mathcal{O}(\Pi \overline{\mathcal{T}}X))$. According to [6] this also implies that the extended moduli space parameterizes A_∞-deformations of $D^b(Coh(X))$ [6]. The solutions of the MC equation

$$\overline{\partial}\gamma + \frac{1}{2}[\gamma, \gamma]_T = 0, \tag{4.44}$$

also parameterizes the consistent deformations of the B model, through the solutions of BV master equation. The path integral of the theory can be viewed as a morphism from $H^*(\texttt{Hoch}(\mathcal{O}(\Pi \overline{\mathcal{T}}X)))$ to $\texttt{Hoch}(\mathcal{O}(\Pi \overline{\mathcal{T}}X))$. A path integral proof that it is a quasi-isomorphism can be done as for the C model. An explicit computation of the path integral will be useful even for very simple manifolds. We expect that both the product and differential of $\mathcal{O}(\Pi \overline{\mathcal{T}}X)$ deform.

Now we make some remarks on adding gauge fields. We regard the target space X as the D-brane world-volume, which has gauge fields on it. The background gauge field can be coupled to the theory by adding the following term to the action

$$\oint_{\partial \Sigma} du^I A_I(u^L). \tag{4.45}$$

For the C model an arbitrary gauge field A can be added without destroying the master equation since $\boldsymbol{Q}\boldsymbol{u}^I = D\boldsymbol{u}^I$ on the boundary. Seiberg-Witten showed that the effective theory is governed by a non-commutative gauge theory, at least for constant B field [73]. For the B model the situation is different, since $\boldsymbol{Q}\boldsymbol{u}^{\bar{i}} = D\boldsymbol{u}^{\bar{i}} + \boldsymbol{\psi}^{\bar{i}}$ on the boundary. Following Witten we consider

$$\oint_{\partial\Sigma} \left(du^I A_I(u^L) - \psi^{\bar{i}} F_{\bar{i}j}(A)\rho^i \right) \tag{4.46}$$

where ρ^i is the 1-form component of $\boldsymbol{u}^i$. Using $\boldsymbol{Q}\psi^{\bar{i}} = 0$ and $\boldsymbol{Q}\rho^i = -du^i$ we see that the above is $\boldsymbol{Q}$-invariant if the $(0,2)$ part of the curvature of the connection 1-form A vanish. In general we can couple connection of $U(N)$ (Hermitian) holomorphic bundles E on X. Similar to the C model case we expect that the D-brane world-volume effective theory is governed by a non-commutative gauge theory due to the bulk term

$$\frac{1}{2} \int_\Sigma d^2\theta \boldsymbol{b}^{ij} \boldsymbol{\chi}_i \boldsymbol{\chi}_j = \frac{1}{2} \int_\Sigma \left(b^{ij} H_i \wedge H_j + \dots \right). \tag{4.47}$$

The above bulk term, after the gauge fixing (4.22), contains

$$\int_\Sigma \frac{1}{2} b_{\overline{ij}} du^{\bar{i}} \wedge du^{\bar{j}} := \int_\Sigma u^*(B^{0,2}) \tag{4.48}$$

where $b_{\overline{ij}} := b^{ij} g_{i\bar{i}} g_{j\bar{j}}$ and $g_{i\bar{j}}$ is a Ricci-flat Kähler metric. The presence of the above term implies, due to the gauge invariance, that the $(0,2)$ part of the curvature 2-form $F^{0,2}$ should be replaced by $F^{0,2} + B^{0,2}$ everywhere in the effective theory.

For X a Calabi-Yau 3-fold, without the bulk deformation by b, Witten determined the effective theory to be the holomorphic Chern-Simons theory,

$$I_{HCS} = \int_X \omega^{3,0} \wedge \mathrm{Tr} \left(A \wedge \overline{\partial} A + \frac{2}{3} A \wedge A \wedge A \right). \tag{4.49}$$

Now by turning on the bulk deformation by b we know that the wedge product among elements of $\Omega^{0,\bullet}(X)$ must be deformed to a suitable non-supercommutative product, say $\star$. We also expect that the differential $\overline{\partial}$ will be deformed in general, say to $\overline{Q}$. Thus the effective theory would look like

$$I = \int_X \omega^{3,0} \wedge \mathrm{Tr} \left(A \star (\overline{Q}A + B^{0,2}\mathbb{I}) + \frac{2}{3} A \star A \star A \right), \tag{4.50}$$

which is more similar to the open string field theory [88].

We may consider arbitrary deformations and more general boundary interactions by including first descendents (1-dimensional observables) of observables. Being inserted in cyclic order in the boundary with punctures, the general correlation functions are no-longer expected to show strict associativity, but will do so only up to homotopy (See a general discussion in [38]). Recall that all the boundary observables originate from $\oplus H^{0,q}(X)$. We also let those 1-dimensional observables take values in $(E_i^\bullet)^* \otimes E_{i+1}^\bullet$, i.e. $Ext(E_i^\bullet, E_{i+1}^\bullet)$. Those may be viewed as the A_∞-category structure on $D^b(Coh(X)$, whose compositions are given by correlation functions.[18]. Being an A_∞-category (as the collection of all A_∞ modules over the A_∞-algebra $(\mathcal{O}(\Pi\overline{\mathcal{T}}X), \overline{\partial}))$ it satisfies a certain MC equation. Now turning on all bulk deformations leads to certain deformations of $D^b(Coh(X))$. The BV master equation, via the general structure of the BV Ward identity, may imply that the deformed category is again an A_∞ category. Since the extended moduli space $\mathcal{M}$ parameterizes all consistent bulk deformations of the extended B-model, we see that $\mathcal{M}$ indeed parameterizes the A_∞ deformations of $D^b(Coh(X))$.

We expect that the above discussions can be made more precise and explicit by carefully studying the extended B-model. We also expect to have explicit results on correlation functions at least for simple cases. The final effective theory will be much more complicated than (4.50), resembling the A_∞-version of open string field theory [29]. In principle the path integral of the extended B-model can lead to explicit answers for all higher compositions (string products). Then it will be the first explicit construction of open string field theory introduced in [29].

4.2.2 Topological open p-branes and generalized homological mirror conjectures

We showed that there is a unified description of A, B and C models based on essentially the same geometrical structure of the topological open membrane. Those models are special limits (or projections or gauge fixing) of the same (ABC) model. More fundamentally we showed that topological open membrane theory can describes and control A, B and C models of topological closed string as different boundary theories. The relation between A and B models are much like different twistings of the same superconformal

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[18] It is rather correlation maps than correlation functions, though we will stick to the original terminology

field theory. Furthermore the topological open membrane theory leads to further (bulk) deformations of the mathematical or physical structure of the boundary string theory. Then it is natural to conjecture that the homological mirror symmetry can be generalized to the category of homotopy 2-algebra.

We may further generalize the topological membrane theory the way we generalized topological strings. The natural candidate is to use essentially the same geometrical structure as the topological open 3-brane. The resulting theory will have a quite large internal symmetry group, and many topological membrane theories in different limits. Then we may use open membrane versions of the resulting theories to state a homotopy 2-algebraic mirror conjecture. Note that the mirror symmetry between the A and B models is a remnant of the duality between type IIA and IIB strings. It would be interesting to examine if the internal symmetry of the generalized topological membrane theory is related to duality in M theory. In general we may consider topological open p-branes on X, based on the p-th iterated superbundle

$$M_p \to (M_{p-1} \to (M_{p-1} \to \ldots \to (M_1 \to X) \ldots).$$

We recall that the total space M_p has a degree $U = p$ symplectic structure ω_p and the base space M_{p-1} is a Lagrangian submanifold with respect to ω_p. The space M_p has various discrete symmetries preserving ω_p. We may take different Lagrangian subspaces and corresponding boundary conditions. Then we may obtain different theories, which can be physically equivalent in a suitable sense. We conjecture that the homological mirror symmetry can be generalized accordingly, at least for Calabi-Yau spaces X. More specifically, we conjecture that for a given homological mirror symmetry we have a corresponding mirror symmetry in the category of p-algebras. Then the mirror symmetry is generalized to physical equivalences of all topological open p-brane theories.

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